Question

a) The function $f(z)$ of the complex variable $z = x + iy$ is differentiable in a domain D. Use the Cauchy-Riemann equations to prove that for z in D,

$$\left(\frac{\partial |f|}{\partial x}\right)^2 + \left(\frac{\partial |f|}{\partial y}\right)^2 = |f'(z)|^2$$

provided $f(z) \neq 0$.

b) Verify that the function $u(x, y) = e^y \cos x$ is harmonic. find a function $v(x, y)$ so that $f = u + iv$ is a differentiable function of the complex variable $z = x + iy$.

Write f as an explicit formula in z.

Answer

a) Let $f = u + iv$. Then $|f|^2 = u^2 + v^2$.

So $2|f|\dfrac{\partial |f|}{\partial x} = 2u\dfrac{\partial u}{\partial x} + 2v\dfrac{\partial v}{\partial x}$

and $2|f|\dfrac{\partial |f|}{\partial y} = 2u\dfrac{\partial u}{\partial y} + 2v\dfrac{\partial v}{\partial y} = -2u\dfrac{\partial v}{\partial x} + 2v\dfrac{\partial u}{\partial x}$

squaring and adding gives

$$|f|^2\left[\left(\frac{\partial |f|}{\partial x}\right)^2 + \left(\frac{\partial |f|}{\partial y}\right)^2\right] = (u^2 + v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$= |f|^2\left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = |f|^2|f'(z)|^2$$

so provided $f(z) \neq 0$, dividing by $|f|^2$ gives the result.

b) $u = e^y \cos x$

So $\dfrac{\partial u}{\partial x} = -e^y \sin x$ and $\dfrac{\partial^2 u}{\partial x^2} = -e^y \cos x$

$\dfrac{\partial u}{\partial y} = e^y \cos x$ and $\dfrac{\partial^2 u}{\partial y^2} = e^y \cos x$

Hence $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ i.e. $u$is harmonic.

Now $\dfrac{\partial u}{\partial x} = -e^y \sin x = \dfrac{\partial v}{\partial y}$, so $v = -e^y \sin x + \phi(x)$

$$\frac{\partial^2 u}{\partial y^2} = e^y \cos x = -\frac{\partial v}{\partial x}, \text{ so } v = -e^y \sin x + \psi(y)$$

Thus $v = -e^y \sin x + C$ (constant)

Now $f = u + iv = e^y(\cos x - i \sin x) + k$

$$= e^y e^{-ix} + k = e^{-i(x+iy)} + k = e^{-iz} + k$$