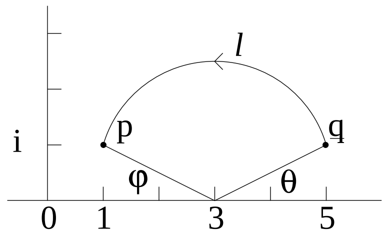
Question

Let $p = 1 + i$ and $q = 5 + i$ be two points in $\mathbf{H}$. Let ℓ be the hyperbolic line segment joining p to q. Determine the hyperbolic length of ℓ.

Answer



(note that $\varphi = \pi - \theta$ since $Im(p) = Im(q)$)

The hyperbolic line segment ℓ joining $p = 1 + i$ to $q = 5 + i$ is contained in the euclidean circle with center 3 and radius 2, and so can be parametrized by

$f : [\theta, \pi - \theta] \longrightarrow \mathbf{H},\ f(t) = 3 + \sqrt{5}e^{it}\ (f(\theta) = q, f(\pi - \theta) = p)$.

$\mathrm{Im(f(t))} = \sqrt{5}\,\mathrm{sin(t)}$

$|f'(t)| = |\sqrt{5}ie^{it}| = \sqrt{5}$

So,

$$\begin{aligned}
\text{length}_{\mathbf{H}}(\text{f}) &= \int_{\theta}^{\pi-\theta} \frac{1}{\sin(t)}\,dt \\
&= \ln|\csc(t) - \cot(t)||_{\theta}^{\pi-\theta} \\
&= \ln\left|\frac{\csc(\pi-\theta) - \cot(\pi-\theta)}{\csc(\theta) - \cot(\theta)}\right| \\
&= \ln\left|\frac{\csc(\theta) + \cot(\theta)}{\csc(\theta) - \cot(\theta)}\right|
\end{aligned}$$

$\sin(\theta) = \frac{1}{\sqrt{5}} \quad \csc(\theta) = \sqrt{5}$
$\cos(\theta) = \frac{2}{\sqrt{5}} \quad \cot(\theta) = 2$

So, $\text{length}_{\mathbf{H}}(f) = \ln\left|\frac{\sqrt{5}+2}{\sqrt{5}-2}\right|$.