Question

Let ℓ be a hyperbolic line in $\mathbf{H}$, and let m and n be elements of $\text{Möb}(\mathbf{H})$ satisfying $m(\ell) = \ell = n(\ell)$. Prove or give a counter-example to the following claim: m and n commute; that is, $m \circ n = n \circ m$.

Answer

Consider ℓ the positive imaginary axis, so that $\text{stab}_{\text{Möb}(\mathbf{H})}(\ell) = G_\ell$ is generated by $\ell_a(z) = az$ $(a > 0), k(z) = \dfrac{-1}{z}$ and $B(z) = -\bar{z}$.

- $\ell_a \circ B(z) = -a\bar{z}$ and $B \circ \ell_a(z) = -a\bar{z}$ (since $a \in \mathbf{R}$) and so ℓ_a, B commute
- $\ell_a \circ k(z) = \dfrac{-a}{z}$ and $k \circ \ell_a(z) = \dfrac{-1}{(az)} \neq \ell_a \circ k(z)$

and so G_ℓ is not a commutative group.