Question
Let E be the ellipse in $\mathbf{C}$ given by the equation

$$E = \left\{ z \in \mathbf{C} \mid \frac{3}{4} \left(\mathrm{Re}(z)\right)^2 + \frac{5}{4} \left(\mathrm{Im}(z)\right)^2 = 1 \right\}.$$

Determine at least three non-trivial elements m of the general Möbius group Möb satisfying $m(E) = E$.

Answer
First, note that E is symmetric with respect to both the x-axis and the y-axis ($\mathbf{R}$-axis and imaginary axis) and so two elements of Möb taking E to E are $C(z) = \bar{z}$ (reflection in $\mathbf{R}$) and $B(z) = -\bar{z}$ (reflection in the imaginary axis). The comparison of B and C is rotation by π fixing $0, \infty$ (i.e. $m(z) = -\bar{z}$), which also takes E to E. So, there is a $\mathbf{Z_2} \oplus \mathbf{Z_2}$ subgroup of Möb generated by $B,\ C$ contained in $G_E = \{m \in \mathrm{Mddotob|m(E) = E}\}$.

[<u>no loxodromic takes E to E</u>: if m is loxodromic and $m(E) = E$, then the fixed points of m are on E. This is probably most easily seen by conjugating so that the fixed points of m are 0 and ∞ and then noting that a curve invariant under such a loxodromic is either a line through 0 or a curve that spirals into 0, and the ellipse gets taken to a curve that does neither.

<u>no parabolic takes E to E</u>; again, if m is parabolic and $m(E) = E$, then the fixed point of m is on E. Conjugating so that $m(z) = z + 1$, note that the curves invariant under m are precisely the horizontal lines and other periodic curves, and the ellipse is neither.

<u>no infinite order elliptic takes E to E</u>: if it did, then E would contain a dense subset of a circle, which would then force E to be a circle, which it isn't.]