

## Exam Question

### Topic: TripleIntegral

A sphere  $S$  whose equation is  $x^2 + y^2 + z^2 = a^2$  has density which is proportional to the square of the distance from the plane  $z + a = 0$ , i.e. the plane which is tangent at the “south pole”. Find the total mass of the sphere.

### Solution

The density is given by  $\rho(x, y, z) = k(a + z)^2$ . The mass is given by

$$M = \iiint_S k(a + z)^2 dV = \iiint_S a^2 dV + 2ka \iiint_S z dV + k \iiint_S z^2 dV.$$

Now  $\iiint_S z dV = 0$  by symmetry.

$$\text{Also } \iiint_S a^2 dV = a^2 \cdot \frac{4}{3}\pi a^3 = \frac{4}{3}\pi a^5.$$

Finally we have, using spherical polar coordinates,

$$\begin{aligned} \iiint_S z^2 dV &= \int_0^{2\pi} d\phi \int_0^\pi d\theta \int_0^a (r \cos \theta)^2 \cdot r^2 \sin \theta dr \\ &= 2\pi \int_0^\pi \sin \theta \cos^2 \theta d\theta \int_0^a r^4 dr \\ &= 2\pi \left[ \frac{\cos^3 \theta}{3} \right]_0^\pi \cdot \frac{a^5}{5} = \frac{4}{15}\pi a^5. \end{aligned}$$