Question
Describe what is meant by a compound Poisson process.

Show that if $A(z)$ is the probability generating function for hte number of events occurring at each point of the process, then the random variable $X()t$ - the total number of events occurring in time t - has probability generating function of the form

$$G_t(z) = \exp(\lambda t A(z) - \lambda t).$$

Suppose that the average number of events occurring at each point of the process if μ. Show that the average number of events occurring in time t is $\mu\lambda t$.

Let the probability that no events occur at a point in the process be p. Find an expression for the probability of no events occurring in time t in the compound process.

Answer
Suppose that

(i) points occur in a Poisson process $\{N(t) : t \geq 0\}$ with rate λ.

(ii) at the ith point Y_i events occur, where $Y_1,\ Y_2, \cdots$ are i.i.d random variables.

(iii) Y_i and $\{N(t) : t \geq 0\}$ are independent.

The total number of events occurring time interval of length t is

$$X(t) = \sum_{i=1}^{N(t)} y_i$$

$\{X(t) : t \geq 0\}$ is said to be a compound Poisson process.
Let the p.g.f of each Y_i be $A(z)$. Then $X(t)$ has p.g.f.

$$\begin{aligned}
& \sum_{j=0}^{\infty} z^j P(X(t)=j) \\
= & \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} z^j P(X(t)=j \mid N(t)=n) P(N(t)=n) \\
= & \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} z^j P(Y_1+\cdots+Y_n=j) \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
= & \sum_{j=0}^{\infty} \left\{ \sum_{n=0}^{\infty} z^j P(Y_1+\cdots+Y_n=j) \right\} \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
= & \sum_{n=0}^{\infty} [A(z)]^n \frac{(\lambda t)^n e^{-\lambda t}}{n!} \text{since the } Y_i \text{ are independent} \\
= & \exp(\lambda t A(z) - \lambda t)
\end{aligned}$$

The average number of events in time t is given by $G'(1)$.

$$G_t'(z) = \exp(\lambda t A(z) - \lambda(t)) \lambda t A'(z)$$

so $G_t'(1) = \exp(\lambda t A(1) - \lambda t) \cdot \lambda t A'(1)$

Now for any p.g.f. $A(z)$, $A(1) = 1$ and $A'(1) = \mu$

so $G_t'(1) = \exp(0) \lambda t A'(10) = \mu \lambda t$.

The probability of no events occurring tiem t is given by $G_t(0) = \exp(\lambda t A(0) - \lambda t)$.

now $A(0) = p$ and so the required probability is $\exp(\lambda t p - \lambda t)$.