Question

Explain what a branching Markov chain is. Suppose a population is descended from a single individual (generation 0). Let $A(s)$ be the probability generating function for the number of offspring of any individual. Let X_n be the number of individuals in general n, with probability generating function $F_n(s)$.

Prove that $F_n(s) = F_{n-1}(A(s))$ and deduce that $F_n(s) = A(F_{n-1}(s))$.

Suppose that the probability distribution of the number Z of offspring of any individual is given by

$$P(Z = k) = qp^k \text{ for k} = 0,\ 1,\ 2, \cdots$$

where $0 < p < 1,\ q = 1 - p$ and $p \neq q$. Obtain the probability generating function $A(s)$ in this case, and verify that for $n = 1,\ 2,\ \cdots$,

$$F_n(s) = \frac{q(p^n - q^n - (p^{n-1} - q^{n-1})ps)}{p^{n+1} - q^{n+1} - (p^n - q^n)ps}$$

Find the probability of eventual extinction of the population.

Answer

Suppose we have a population of individuals, each reproducing independently of the others. Suppose the distributions of the number of offspring of all individuals are identical. Let X_n be the number of individuals in the nth generation. Then (X_n) is a branching Markov chain.

Suppose $P(z = k) = a_k$ and $A(s) = \sum_{k=0}^{\infty} a_k s^k$.

Now $P(X_n = l \mid X_{n-1} = j) = P(z_1 + \cdots + z_j = 1) =$ coeff. of s^l in $[A(s)]^j$ as the z_i are i.i.d.

$P(x_n = l) = \sum_{j=0}^{\infty} P(X_n = 1 \mid X_{n-1} = j) P(X_{n-1} = j)$

so

$$\begin{aligned}
F_n(s) &= \sum_{l=0}^{\infty}\sum_{j=0}^{\infty} (\text{coeff. of } s^l \text{ in } [A(s)]^j) P(X_{n-1} = j) s^l \\
&= \sum_{l=0}^{\infty} \left(\sum_{j=0}^{\infty} (\text{coeff. of } s^l \text{ in } [A(s)]^j) s^l \right) P(X_{n-1} = j) \\
&= \sum_{j=0}^{\infty} P(X_{n-1} = j)[A(s)]^j = F_{n-1}(A(s))
\end{aligned}$$

Now

$P(X_0 = 1) = 1$ so $F_0(s) = s$.

$F_1(s) = F_0(A(s)) = A(s)$
$F_2(s) = F_1(A(s)) = A(A(s))$
$\vdots$
$F_n(s) = \underbrace{A(A(\ \cdots\ (A}_{n \text{ times}}(s)))\cdots = A(F_{n-1}(s))$

Now when $a_k = qp^k$

$$A(s) = \sum_{k=0}^{\infty} qp^k s^k = \frac{q}{1-p^s}$$

Now $F_1(s) = A(s)$ - which fits the given formula for $n = 1$. Assume the formula is true for n

$$\begin{aligned}
& F_{n+1}(s) \\
= \; & A(F_n(s)) \\
= \; & \frac{q}{1 - \frac{pq[p^n - q^n - (p^{n-1} - q^{n-1})ps]}{p^{n+1} - q^{n+1} - (p^n - q^n)ps}} \\
= \; & \frac{q[p^{n+1} - q^{n+1} - (p^n - q^n)ps]}{p^{n+1} - q^{n+1} - (p^n - q^n)ps - pq[p^n - q^n - (p^{n-1} - q^{n-1})ps]} \\
= \; & \frac{q[p^{n+1} - q^{n+1} - (p^n - q^n)ps]}{p^{n+1}(1-q) - q^{n+1}(1-p) - [p^n(1-q) - q^n(1-p)]ps} \\
= \; & \frac{q[p^{n+1} - q^{n+1} - (p^n - q^n)ps]}{p^{n+2} - q^{n+2} - (p^{n+1} - q^{n+1})ps}
\end{aligned}$$

as $p + q = 1$.
Hence the result by induction.
The probability of extinction is the smallest positive root of the equation $A(s) = s$, and so is given by

$$\frac{q}{1-ps} = s$$

i.e. $ps^2 - s + q = 0 \quad (ps - q)(s - 1) = 0$ as $p + q = 1$
$s = \frac{q}{p},\ s = 1$
So the extinction probability is 1 if $q \geq p$ and $\frac{q}{p}$ if $q < p$.