Question

(a) A Markov chain has three states, and transition probability matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1-p & 0 & p \\ 0 & 1 & 0 \end{pmatrix}$$

where $0 < p < 1$. Find the probability distribution for state occupancy at the nth step ($n \geq 1$) if initially all the states are equally likely to be occupied.

(b) A Markov chain has the transition probability matrix given below. Classify the states and find the mean recurrence times for all recurrent states. (Label the states 1,2,3,4,5,6,7 in order)

$$P = \begin{pmatrix} 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{8} & \frac{1}{8} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Answer

(a) $P^2 = \begin{pmatrix} 1-p & 0 & p \\ 0 & 1 & 0 \\ 1-p & 0 & p \end{pmatrix}$ and $P^3 = P$

so $P^n = P$ if n is odd and $P^n = P^2$ if n is even. So if the initial distribution is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ the distribution at step n is:

$\left(\frac{1}{3}(1-p), \frac{2}{3}, \frac{1}{3}p\right)$ if n is odd.

$\left(\frac{2}{3}(1-p), \frac{1}{3}, \frac{2}{3}p\right)$ if n is even.

(b) The transition diagram is as follows:

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$\{2,3\}$ and $\{4,6\}$ both form irreducible closed aperiodic sets of states, and so are ergodic.

States 1, 5 intercommunicate and are of the same type.

The probability of return to state 1 is

$$\begin{aligned} f_{11} &= \frac{1}{8}\cdot\frac{1}{3}+\frac{1}{8}\cdot\frac{1}{6}\cdot\frac{1}{3}+\frac{1}{8}\cdot\left(\frac{1}{6}\right)^2\cdot\frac{1}{3}+\cdots \\ &= \frac{1}{8}\cdot\frac{1}{3}\left(1+\frac{1}{6}+\frac{1}{6^2}+\cdots\right) \\ &= \frac{1}{8}\cdot\frac{1}{3}\cdot\frac{1}{1-\frac{1}{6}} \\ &= \frac{1}{8}\cdot\frac{1}{3}\cdot\frac{6}{5}=\frac{1}{20}<1 \end{aligned}$$

Hence states 1 and 5 are transient.

OR:

The probability of leaving state 1 initially to a state other than 5 is $\frac{7}{8}$. Return is only possible from state 5. So the probability of return is at most $\frac{1}{8}<1$.

State 7 is transient since the probability of return is zero.

To calculate mean recurrence times

$$\begin{aligned} \mu_2 &= \frac{1}{4}+2\cdot\frac{3}{4}\cdot\frac{1}{3}+3\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{3}+4\cdot\frac{3}{4}\cdot\left(\frac{2}{3}\right)^2\cdot\frac{1}{3}+\cdots \\ &= \frac{1}{4}+\frac{3}{4}\cdot\frac{1}{3}\left[2+3\cdot\frac{2}{3}+4\left(\frac{2}{3}\right)^2+\cdots\right] \end{aligned}$$

$$\begin{aligned} s &= a + (a+1)r + (a+2)r^2 + \cdots \\ rs &= \qquad ar + (a+1)r^2 + \cdots \\ s-rs &= a + r + r^2 + \cdots \\ &= a + \frac{r}{1-r} \end{aligned}$$

$$s=\frac{a}{1-r}+\frac{r}{(1-r)^2}$$

so $\mu_2=\frac{1}{4}+\frac{3}{4}\cdot\frac{1}{3}\left[3\cdot 2+9\cdot\frac{2}{3}\right]=3\frac{1}{4}$

$$
\begin{aligned}
\mu_3 &= \frac{2}{3} + \frac{1}{3} \cdot \frac{3}{4} \left[2 + 3 \cdot \frac{1}{4} + 4 \cdot \left(\frac{1}{4} \right)^2 + \cdots \right] \\
&= \frac{2}{3} + \frac{1}{3} \cdot \frac{3}{4} \left[2 \cdot \frac{4}{3} + \frac{1}{4} \cdot \left(\frac{4}{3} \right)^2 \right] \\
&= \frac{2}{3} + \left(\frac{2}{3} + \frac{1}{9} \right] \\
&= 1\frac{4}{9}
\end{aligned}
$$

$$
\begin{aligned}
\mu_4 &= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} \left[2 + 3 \cdot \frac{3}{4} + 4 \cdot \left(\frac{3}{4} \right)^2 + \cdots \right] \\
&= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} \left[2 \cdot 4 + \frac{3}{4} \cdot 4^2 \right] \\
&= \frac{1}{2} + \frac{1}{2} \cdot 5 = 3
\end{aligned}
$$

$$
\begin{aligned}
\mu_6 &= \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{4} \left[2 + 3 \cdot \frac{1}{2} + 4 \cdot \left(\frac{1}{2} \right)^2 + \cdots \right] \\
&= \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{4} \left[2 \cdot 2 + \frac{1}{2} \cdot 2^2 \right] \\
&= \frac{3}{4} + \left(\frac{1}{4} + 3 \right] \\
&= \frac{3}{2}
\end{aligned}
$$