Question

A simple random walk can occupy states $(0,\ 1,\ 2,\ \cdots)$. For states $j \geq 2$ there is a probability p of making a step of +1, and q of making a step of -1, where $p + q = 1$ and $pq \neq 0$. State 0 is an absorbing barrier. State 1 is partially reflecting in the sense of the following conditional probabilities:

$$\begin{aligned} P(X_{n+1} = 2 \mid X_n = 1) &= p \\ P(X_{n+1} = 1 \mid X_n = 1) &= qk \\ P(X_{n+1} = 0 \mid X_n = 1) &= p(1-k) \end{aligned}$$

where $0 < k < 1$.

Let $f(n,\ j)$ denote the probability of absorption at step n, starting in state j. Let $F_j(s)$ denote the generating function for the sequence of probabilities $f(n, j)$ $(n = 0,\ 1,\ 2,\ \cdots)$.
Explain why $F_j(1)$ gives the probability of eventual absorption, starting in state j.

Write down a difference equation for $f(n, j)$, by arguing conditionally on the result of the first step, for $j = 1$ and for $j > 1$.

Hence obtain a difference equation for $F_j(1)$, the probability of eventual absorption, for $j = 1$ and for $j > 1$. Find the general solution of the difference equation for $j > 1$. Show by considering large j, that for $q > p$ and for $q = p$, $F_j(1)$ is constant,a nd further that $F_j(1) = 1$.

For $q > p$, using the assumption that $F_j(1) \to 0$ as $j \to \infty$, show that

$$F_j(1) = \frac{p(1-k)}{p-kq}\left(\frac{q}{p}\right)^j \text{ orj} > 0.$$

Answer

$F_j(s) = \sum_{n=0}^{\infty} f(n,\ j)s^n$

$F_j(1) = \sum_{n=0}^{\infty} f(n,\ j)$

which is the probability of absorption, either at step 0, or step 1, or step 2, $\cdots$, i.e. the probability of eventual absorption.

Now arguing conditionally on the result of the first step, for $j = 1$, gives

$f(n,\ 1) = pf(n-1,\ 2) + q(1-k)f(n-1,\ 0) + qkf(n-1,\ 1)\ \ (A)$

For $j > 1$ we obtain

$f(n,\ j) = pf(n-1,\ j+1) + qf(n-1,\ j-1) \quad (B)$

$$\begin{aligned} \text{Now} \qquad f(0,1) &= 0 \\ f(n,\ 0) &= \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \\ \text{so} \qquad f(n-1,\ 0) &= \begin{cases} 1 & n = 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Summing (A) for $n = 1,\ 2,\ \cdots$ gives

$$F(1) = p\sum_{n=1}^{\infty} f(n-1,\ 2) + q(1-k) + qk\sum_{n=1}^{\infty} f(n-1,\ 1)$$

$F_1(1) = pF_2(1) + q(1-k) + qkF_1(1) \quad (C)$

For $j > 1,\ f(0,\ j) = 0$, so summing (B) for $n = 1,\ 2,\ \cdots$ gives

$$F_j(1) = pF_{j+1}(1) + qF_{j-1}(1)$$

Now let $F_j(1) = \lambda^j \quad j \geq 1$

then $p\lambda^2 - \lambda + q = 0$, so $(p\lambda - q)(\lambda - 1) = 0$.

$$\text{For} \quad \begin{aligned} &p \neq q \quad F_j(1) = A\left(\frac{q}{p}\right)^j + B \\ &p = q \quad F_j(1) = Aj + B \end{aligned}$$

Now if $q > p$ or $q = p,\ A \neq C \rightarrow F_j(1)$ is outside $[0,1]$ for large j, and so couldn't represent a probability. Hence $A = C$ and so for $q \geq p,\ F_j(1) = B$.

Using (C) now gives

$B = pB + q(1-k) + qkB$

$B(1 - p - qk) = q - qk$

$B = 1$ (as $p + q = 1$ and $0 < k < 1$)

Hence $F_j(1) = 1$.

Now if $q < p$, $\left(\frac{q}{p}\right)^j \rightarrow 0$ as $j \rightarrow \infty$ so assuming $F_j(1) \rightarrow 0$ this gives $B = C$.

Hence $F_j(1) = A\left(\frac{q}{p}\right)^j$ and using (C) again gives

$$A\left(\frac{q}{p}\right) = pA\left(\frac{q}{p}\right)^2 + q(1-k) + qkA\left(\frac{q}{p}\right)$$

$$A\left(\frac{q}{p} - \frac{q^2}{p} - \frac{q \cdot 2 \cdot k}{p}\right) = q(1-k)$$

so $A = \dfrac{p(1-k)}{p - qk}$ using $P + q = 1$.

Thus $F_j(1) = \dfrac{p(1-k)}{p - kq}\left(\dfrac{q}{p}\right)^j$ in this case.