Question

(a) A gambler with initial capital z plays against an opponent with initial capital $(a-z)$ where a and z are integers, and $0 \leq z \leq a$. At each play the gambler wins 1 with probability $\frac{1}{3}$, loses 1 with probability $\frac{1}{2}$, and retains his stake money in the event of a draw.

Formulate a difference equation for the probability P_z that the game ends in an even number of bets, with appropriate boundary conditions involving P_0 and P_a. Solve the equation to find an explicit formula for P_z in terms of z and a.

What is the value of P_z if the gambler plays against an infinitely rich opponent?

(b) In the classical gambler's ruin problem one of the players will eventually be ruined, with probability 1. Write a brief explanation of what is meant by thus, giving particular attention to the concept "with probability 1" in this context.

Answer

(a) Consider the first bet. There are 3 possible outcomes

(i) Gambler wins. Then he has $z+1$ and the game must not end in a further even number of bets. Prob: $\frac{1}{3}(1-P_{z+1})$.

(ii) Gambler loses. Then he has $z-1$ and the game must not end in a further even number of bets. Prob: $\frac{1}{2}(1-P_{z-1})$.

(iii) Draw. Then the gambler still has z and the game must not end in a further even number of bets. Prob: $\frac{1}{6}(1-P_z)$.

So $P_z = \frac{1}{3}(1-P_{z+1}) + \frac{1}{2}(1-P_{z-1}) + \frac{1}{6}(1-P_z)$ for $0 < z < a$. $P_0 = 1,\ P_a = 1$

The equation when simplified becomes

$$2P_{z+1} + 7P_z + 3P_{z-1} = 6 \quad 0 < z < a$$

Putting $P_z = \lambda^2$ in the homogeneous equation gives

$$2\lambda^2 + 7\lambda + 3 = 0 \quad (2\lambda + 1)(\lambda + 3) = 0$$

so $\lambda = -\frac{1}{2}, \ -3$.
A particular solution of the inhomogeneous equation is
$P_z = \frac{1}{z}(\cos t)$????????

So the general solution is $P_z = A\left(-\frac{1}{2}\right)^2 + B(-3)^2 + \frac{1}{2}$

The boundary conditions give $\begin{array}{l} P_0 = A + B + \frac{1}{2} = 1 \\ p_a = A\left(-\frac{1}{2}\right)^a + B(-3)^a + \frac{1}{2} = 1 \end{array}$

Solving these gives

$$B = \frac{1}{2}\frac{((-2)^a - 1)}{6^a - 1} \quad A = -B + \frac{1}{2}$$

$$P_z = \frac{1}{2}\frac{((-2)^a - 1)}{6^a - 1}\left[(-3)^z - \left(-\frac{1}{2}\right)^2\right] + \frac{1}{2}\left[1 + \left(-\frac{1}{2}\right)^z\right]$$

As $a \to \infty$ $\;P_z \to \frac{1}{2}\left(+\left(-\frac{1}{2}\right)^z\right)$

(b) Consider a large number N of realisations of the procedure, and let $R(N)$ denote the number of these which result in one of other of the players being ruined. Let $D(N) = N - R(N)$. Then to say that one of the players will be ruined, with probability 1, means that $\dfrac{R(N)}{N} \to 1$ as $N \to \infty$.

It also implies $\dfrac{D(N)}{N} \to 0$ as $N \to \infty$ so that neither player is ruined, with probability zero. This does not mean impossibility, as clearly the two players could win alternately and the game would continue for ever. Thus "with probability 1" does not mean certainty.