Question

State Lebesgue's theorem on dominated convergence for a sequence of functions $\{f_n\}$.

The functions $f(x,t)$ has the following properties:

i) $f(x,t)$ is an integrable function of x, for each t;

ii) there is a function $h(x)$ such that for each x, $\lim_{t \to a} f(x,t) = h(x)$, where a is a fixed real number;

iii) there is an integrable function $g(x)$ with the property that, for each x, there exists $\delta > 0$ such that for $|t-a| < \delta$,

$$|f(x,t)| \le g(x).$$

Use Lebesgue's theorem to show that

$$\lim_{t \to a} \int f(x,t)dx = \int h(x)dx.$$

Suppose that $\phi(x,t)$ is an integrable function of x, for each t, and that the partial derivative $\frac{\partial \phi}{\partial t}$ exists for all x, t.

Suppose also that there is an integrable function $\psi(x) > 0$ such that

$$\left| \frac{\partial \phi}{\partial t} \right| \le \psi(x)$$

for all x. Prove that

$$\frac{d}{dt} \int \phi(x,t)dx = \int \frac{\partial \phi}{\partial t} dx.$$

If $I(\alpha) = \int_0^\infty x^\alpha e^{-x} dx, \qquad \alpha > 0,$

show that

$$I(\alpha) = I'(\alpha+1) - (\alpha+1)I'(\alpha).$$

Answer

Lebesgue's theorem on dominated convergence states that if $\{f_n\}$ is a sequence of measurable functions with the property that $|f_n(x)| \leq g(x)$ for all n, x where $g(x)$ is integrable, and if $f_n(x) \to f(x)$ as $n \to \infty$ for all x, then f is integrable, and $\lim\limits_{n\to\infty} \int f_n(x)dx = \int f(x)dx$.

Let $\{t_n(x)\}$ be an arbitrary sequence of real numbers satisfying $|t_n - a| < \delta(x)$ and converging to a.

Let $f_n(x) = f(x, t_n(x))$

Then $|f_n(x)| \leq g(x)$ for all x.

Thus $\lim\limits_{n\to\infty} \int f(x, t_n)dx = \int h(x)dx$ by Lebesgue's theorem.

Hence $\lim\limits_{t\to\infty} \int f(x,t)dx = \int h(x)dx$

Now let $\Phi(t) = \int \phi(x,t)dx$

Consider $\dfrac{\Phi(t+h) - \Phi(t)}{h}$

$= \displaystyle\int \frac{\phi(x,t+h) - \phi(x,t)}{h} dx = \int X(x,h)dx$

Now $X(x,h) \to \left.\dfrac{\partial\phi}{\partial t}\right|_x$ as $h \to 0$

Thus there exists δ such that for all h satisfying $0 < |h| < \delta$,

$\left| X(x,h) - \left.\dfrac{\partial\phi}{\partial t}\right|_x \right| < \psi(x)$

Thus for all h satisfying $0 < |h| < \delta$, we have

$|X(x,h)| \leq \left| \left.\dfrac{\partial\phi}{\partial t}\right|_x \right| + \psi(x) \leq 2\psi(x)$

Since ψ is integrable 2ψ is integrable and so the above result shows that

$\lim\limits_{h\to 0} \displaystyle\int X(x,h)dx = \int \frac{\partial\phi}{\partial t} dx$

i.e. $\dfrac{d}{dt} \displaystyle\int \phi(x,t)dx = \int \frac{\partial\phi}{\partial t} dx$

$I(\alpha) = \displaystyle\int_0^\infty x^\alpha e^{-x} dx$

$\dfrac{\partial}{\partial\alpha}(x^\alpha e^{-x}) = x^\alpha (\log x) e^{-x}$

$= x^\alpha (\log x) e^{-\frac{1}{2}x} e^{-\frac{1}{2}x} < e^{-\frac{1}{2}x}$ for $x \geq x_0$

since $x^\alpha (\log x) e^{-\frac{1}{2}x} \to 0$ as $x \to \infty$

also $x^\alpha (\log x) e^{-\frac{1}{2}x} \to 0$ as $x \to 0$ and so

$|x^\alpha (\log x) e^{-\frac{1}{2}x}| < 1$ if $0 < x \le x_1$.

Thus if

$$\psi(x) = \begin{cases} 1 & 0 < x \le x_1 \\ |x^\alpha (\log x) e^{-\frac{1}{2}x}| & x_1 < x < x_2 \\ e^{-\frac{1}{2}x} & x \ge x_2 \end{cases}$$

$\left| \dfrac{\partial}{\partial \alpha}(x^\alpha e^{-x}) \right| \le \psi(x)$ and ψ is integrable,

thus $I'(\alpha) = \displaystyle\int_0^\infty \frac{\partial}{\partial \alpha}(x^\alpha e^{-x})$

$$(\alpha+1)I'(\alpha) = \int_0^\infty (\alpha+1)x^\alpha (\log x) e^{-x} dx$$

$$= \left[x^{\alpha+1}(\log x)e^{-x}\right]_0^\infty - \int_0^\infty x^{\alpha+1}\left[e^{-x}\frac{1}{x} - (\log x)e^{-x}\right] dx$$

$$= \int_0^\infty x^{\alpha+1}(\log x)e^{-x}dx - \int_0^\infty x^\alpha e^{-x} dx$$

$$= I'(\alpha+1) - I(\alpha)$$

Hence $I(\alpha) = I'(\alpha+1) - (\alpha+1)I'(\alpha)$