Question

Discuss Riemann and Lebsgue integrability on (0,1) of the functions f and g defined below. In each case calculate the integrals, where they exist.

(a) $f(x) = 0$ if x is irrational,

$f(x) = \frac{1}{c}$ if x is rational and $x = \frac{b}{c}$ where b and c have no common factors.

(b) $g(x) = 0$ if x is rational,

$g(x) = \frac{1}{a}$ if x is irrational, where a is the first non -zero integer in the decimal representation of x.

(In each case you should prove any assertions you make concerning continuity of function. Conditions for integrability should be stated but not proved.)

Answer

(a)

$$\begin{aligned} f(x) &= 0 \quad x \text{ rational} \\ f(x) &= \frac{1}{c} \quad x \text{ is rational and } x = \frac{b}{c} \end{aligned}$$

where b and c have no common factors.

We prove that f is continious at each irrational point. Let $x\epsilon(0,1)$ be irrational. Let $\epsilon > 0$ be given. Choose $n > \frac{1}{\epsilon}$.

Since x is irrational, there is an integer m such that

$$\frac{m}{n!} < x < \frac{m+1}{n!} \qquad 0 \leq m \leq n!$$

Let $\delta = min\left\{x - \frac{m}{n!}, \frac{m+1}{n!} - x\right\}$

Consider the interval $I = (x - \delta,\, x + \delta)$

If $y\epsilon I$ and y is irrational $|f(y) - f(x)| = 0 < \epsilon$

If $y\epsilon I$ and y is rational then $y = \frac{b}{c}$, where $(b, c) = 1$ and $\underline{c > n}$. Hence $|f(y) - |f(x)| = \frac{1}{c} < \frac{1}{n} < \epsilon$.

Thus for all $y\epsilon(x - \delta,\, x + \delta)$, $|f(y) - f(x)| < \epsilon$ and so f is continuous at x.

Thus f is continuous almost everywhere and so is R - integrable. Thus f is also l-integrable and

$$R\int_0^1 f = L\int_0^1 f.$$

Now $f = 0$ a.e. and so

$$(L)\int_0^1 f - (L)\int_0^1 o = 0$$

Hence

$$(R)\int_0^1 f = (L)\int_0^1 f = 0$$

(b)

$$\begin{aligned} g(x) &= 0 && x \text{ irrational} \\ g(x) &= \frac{1}{a} && x \text{ is irrational and where } a \text{ isthefirst nonzero} \\ & && \text{didget in the decimal representation of } x \end{aligned}$$

We prove that g is discontinuous at all irrational points.

Let $x\epsilon(0,1)$ be irrational, and $g(x) = \frac{1}{a} > 0$.

Let $\epsilon = \frac{1}{2a}$

Then for each $\delta > 0$ there is a rational y satisfying $|y - x| = \delta$ and so $|g(y) - g(x)| = \frac{1}{a} > \epsilon$.

Hence g is discontinuous at x.

Thus the set of discontinuations of g in [0,1] form a set of measure 1, and so g is not Riemann integrable.

We now prove that g is a simple function, and so is Lebesgue integrable.

g takes only the values $0, \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{1}$.

$\{x|g(x) = 0\} = [0,1] \cap \mathbf{Q}$ and so has measure zero, and is this measurable? ($\mathbf{Q}$ = set of rational numbers)

$\left\{x|g(x) = \frac{1}{a}\right\} = \bigcup_{n=1}^{\infty} \left\{x|\frac{a}{10^n} \leq x \leq \frac{a+1}{10^n}\right\} \cap C(Q)$

and this this measurable, being the union of a countable collection of measurable sets intersected with a measurable set.

Also, since $m(Q) = 0$,

$$\begin{aligned} m\left(\left\{x|g(x) = \frac{1}{a}\right\}\right) &= m\left(\bigcup_{n=1}^{\infty}\left\{x|\frac{a}{10^n} \leq x \leq \frac{a+1}{10^n}\right\}\right) \\ &= \sum_{n=1}^{\infty} m\left(\left\{x|\frac{a}{10^n} \leq x \leq \frac{a+1}{10^n}\right\}\right) \\ &= \sum_{n=1}^{\infty}\frac{1}{10^n} = \frac{1}{9} \end{aligned}$$

Thus g is a function of the form

$$g(x) = \sum_{i=1}^{k} c_i X_{Ei}(x)$$

and so

$$(L)\int_0^1 g = \sum c_i m(E_i) = \frac{1}{9}\left(\frac{1}{9}+\frac{1}{8}+\frac{1}{7}+\frac{1}{6}+\frac{1}{5}+\frac{1}{4}+\frac{1}{3}+\frac{1}{2}+1\right)$$

$$\left(= \frac{7129}{22680} = 0.314\right)$$