Question
Give an outline of the development necessary for a definition of the Lebesgue intergral of measurable function $f : \mathbf{R}^n \to \mathbf{R}$. Any theorems concerning measurable functions should be stated but not proved.
Show that if f and g are two integrable functions then the function $min(f,g)$ is integratable, and that

$$\int min(f,g) \leq min\left(\int f, \int g\right).$$

Discuss the case when equality occurs.

Answer
We first consider the so called simple function. A function $f : \Omega \to R^*$ is called a simple function if it can be expressed in the form

$$f(x) \quad = \quad \sum_{i=1}^{n} c_i X_{Ei}(x) \tag{1}$$

where $e_i \epsilon R^*$, $\{E_1, E_2, \ldots E_n\}$ is a partition of $Omega$ into disjoint measurable sets, and $X_A(x)$ is the characteristic function of the set A.
We then introduce the concept of a measurable function.. A function $f; \Omega \to R^*$ is said to be measurable if and onlt if for all $c \epsilon R^*$, $\{x | f(x) \leq c\}$ is a measurable set.
We the prove the fundemental result stating that any non-negitative measurable function can be expressed as the limit if monotonic increasing sequence of simple functions of the form (1) by

$$\int f = \sum_{i=1}^{n} c_i \cdot m(E_i),$$

proving htat $\int f$ is independent of the representation of f in the form (1).
We then define the integral of a non-negative measurable function f by expressing f as

$$f = \lim_{n \to \infty} f_n \tag{2}$$

where $\{f_n\}$ is an increasing sequence of simple functions, and by defining

$$\int f = \lim_{n \to \infty} \int f_n,$$

proving also that $\int f$ independent of the representation of f in the form (2)

We extend the definition to measurable functions which are positive or negative by adding the functions $f_+,\ f_-$:

$$f_+(x) = \begin{cases} f(x) & \text{if} \quad f(x) > 0 \\ 0 & \text{if} \quad f(x) \le 0 \end{cases}$$

$$F_-(x) = \begin{cases} -f(x) & \text{if} \quad f(x) < 0 \\ 0 & \text{if} \quad f(x) \ge 0 \end{cases}$$

Then for all x we have that $f(x) = f_+(x) - f_-(x)$. After proving that for f, measurable, f_+ and f_- are also measurable, we define

$$\int f = \int f_+ - \int f_-$$

whenever R.H.S. is meaningful (i.e.excluding $\infty - \infty$)

We now prove that

$$min(f,g) = f - (f-g)_+$$

Suppose $f(x) \le g(x)$ then $f(x) - g(x) \le 0$ So $(f-g)_+(x) = 0$ and $min(f(x), g(x)) = f(x)$ If $f(x) > g(x)$ then $f(x) - g(x) > 0$. Thus $f(x) - (f-g)_+(x) = g(x) = min(f(x), g(x))$.
Now the difference of two integrable function is integrable, and so

$$min(f,g) = f - (f-g)_+$$

is integrable. Also

$$\int min(f,g) = \int f - \int (f-g)_+ \le \int f$$

since $(f-g)_+ \ge 0$ and so $\int (f-g)_+ \ge 0$
Similarly

$$\int min(f,g) \le \int g,$$

and so

$$\int min(f,g) \le min\left(\int f \ \int g\right).$$

Suppose $min\left(\int f \ \int g\right) = \int f$ and that $\int min(f,g) = \int f$ then $\int (f-g)_+ = 0$. But $(f-g)_+ \ge 0$ so $(f-g)_+ = 0$ a.e.
Therefore $f(x) \le g(x)$ a.e.