Question

Define what is meant by a measurable set, and what is meant by a measurable function.

Show that if $\{A_n\}$ is a sequence of measurable sets with the properties $A_{n+1} \subseteq A_n$ for$n = 1, 2, \cdots$ and

$$m(A_1) < \infty$$

then

$$m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n).$$

Suppose f is a measurable function defined on [0,1]. Define the function $g(x)$ by

$$g(x) = m(\{y : f(y) \geq x\}).$$

Show that for each real number a,

$$\lim_{x\to a-} g(x) = g(a).$$

Is it true

$$\lim_{x\to a+} g(x) = g(a)?$$

Justify your assertion.

Answer

A set E is said to be measurable if for each set S, we have

$$m^*(S) = m^*(S - E) + m^*(S \cap E)$$

A function F is said to be measurable if for each number c, the set

$$\{x|f(x) \geq c\}$$

is measurable.

We first prove that if A_1, A_2 are all measurable ad $A - 1 \subseteq A_2 \subseteq \cdots$ then

$$\begin{aligned} m\left(\bigcup_{n=1}^{\infty} A_n\right) &= \lim_{n\to\infty} m(A_n) \\ \bigcup_{n=1}^{\infty} A_n &= A_1 \cup (A_2 - A_1) \cup (A_3 - A_2) \cup \cdots \\ &= A_1 \cup \bigcup_{n=1}^{\infty} (A_{n+1} - A_n) \end{aligned}$$

So, by additivity ,

$$\begin{aligned} m\left(\bigcup_{n=1}^{\infty}\right) &= m(A_i) + \sum_{n-1}^{\infty} m(A_{n+1} - A_n) \\ &= \lim_{n\to\infty}[m(A_1) + (A_2 - A_n) + \cdots + (A_{n+1} - A_n)] \\ &= \lim_{n\to\infty} m(A_n) \end{aligned}$$

Now let $B_i = A_1 - A_i$

$\phi = B_1 \subseteq B_2 \subseteq \cdots$ so $m\left(\bigcup_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} m(B_n)$

Thus $m\left(A_1 - \bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} m(A_1 - A_n)$

Therefore $m(A_1) - m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty}(m(A-1) - m(A-n))$

using $m(A_i) < \infty$

Therefore $m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n)$

Let $g(x) = m(\{y|f(y) \geq x\})$

Let $a_1, a_2, \ldots$ be an arbitary sequence with the properties that $a_1 \leq a_2 \leq \ldots$ and $a_n \to a$

Let $A_n = \{y|f(y) \geq a_n\}$

Let $A = \{y|f(y) \geq a\}$

We first prove that $A = \bigcap_{n=1}^{\infty} A_n$

Suppose $y\epsilon A$ then $f(y) \geq a$ and so fo all n, $f(y) \geq a_n$. Hence $y\epsilon \bigcap_{n=1}^{\infty} A_n$ conversley if $y\epsilon \bigcap_{n=1}^{\infty} A_n$ then for all n, $f(y) \geq a_n$ and so $f(y) \geq a_{n+1} \geq a_n$ and so $y\epsilon A_n$.

Also , since $A_1 \subseteq [0,1]$, we have $m(A_1) \leq 1 \leq \infty$.

Hence $m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n)$

or $g(a) = \lim_{n\to\infty} g(a_n)$

But a_n is an arbitary increasing sequence $\to a$ and so $g(a) = \lim_{x\to a-} g(x)$

It is not true that $\lim_{x\to a+} g(x) = g(a)$ as the following example shows:

Let $f(x) = 1$ for all $x\epsilon[0,1]$

$g(1) = m(\{y|f(y) \geq 1\}) = 1$

If $x > 1$ $g(x) = m(\{y|f(y) \geq x > 1\}) = 0$

So $\lim_{x\to 1+} g(x) = 0 \neq g(1)$