**Question**
Let $A$ be a closed measurable set of real numbers. Prove that if $A$ has an interior point then $A$ has positive measure. Is the converse true? Justify your answer fully.
(Any properties of Lebesgue measure used should be stated explicitly.)

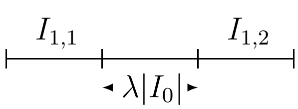
**Answer**
Suppose $A$ has an interior point $a$.
Then there exists $\epsilon > 0$ such that $(a-\epsilon, a+\epsilon) \subseteq A$

$$m((a-\epsilon, a+\epsilon)) = 2\epsilon > 0,$$

and so, since if $S \subseteq T$, $m(S) \leq m(T)$

$$m(A) \geq 2\epsilon > 0$$

The converse if NOT true as the following examples show.
Let $I_0$ be the unit interval. Delete the middle open interval of length $\lambda_1|I_0|$. There remain tow closed interval, denoted by $I_{1,1}$ and $I_{1,2}$ of length $\frac{1}{2}(1-\lambda_1)$ each.




Let $S_1 = \bigcup_{1=2}^{2} I_{1,i}$
We delete from $I_{11}$ and $I_{12}$ the middle open interval of length $\lambda_2|I_{11}|$


There remain four closed intervals, denoted by $I_{2,1}$, $I_{2,2}$, $I_{2,3}$ and $I_{2,4}$ each of length $\frac{1}{2^2}(1-\lambda_1)(1-\lambda_2)$
Let $S_2 = \bigcup_{i=1}^{4} I_{2,i}$
We proceed inductively.
Let $S_n = \bigcup_{i=1}^{2^n} I_{n,i}$ where $|I_{n,i}| = \frac{1}{2^n}\prod_{j=1}^{n}(1-\lambda j)$, $i = 1, 2, \cdots 2^n; n = 1, 2, \cdots k$

We then remove the middle open interval of length $\lambda_{k+1}|I_{k,i}|$ from each $I_{k,i}$. This leaves $2^{k+1}$ closed intervals $I_{k+1,i}$, $i = 1, 2, ..., 2^{k+1}$ each of length $\frac{1}{2^{k+1}} \prod_{j=1}^{k+1} (1 = \lambda_j)$.

Let $s_{k+1} = \bigcup_{i=1}^{2^{k+1}} I_{k+1,i}$ Since, for any closed interval, $m(I) = |I|$.and since $m$ is additive over disjoint sets, we have

$$\begin{aligned} m(S_k) &= \sum_{i=1}^{2^k} m(I_{k,i}) \\ &= \sum_{i=1}^{2^k} |I_{k,i}| \\ &= \sum_{i=1}^{2^k} \frac{1}{2^k} \prod_{j=1}^{k} (1 - \lambda_j) \\ &= \prod_{j=1}^{k} (1 - \lambda_j) \end{aligned}$$

Also $m(I_0) = 1 < \infty$, and $i_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$

Hence if we let $S = \bigcap_{k=1}^{\infty} S_k$ $S$ is a non-empty closed set and therefore measurable, and we have $m(s) = \lim_{k \to \infty} m(s_k) = \prod_{j=1}^{\infty} (1 - \lambda_j)$ we choose $\lambda_j = \frac{1}{2^j}$. Then since $\sum \frac{1}{2^j}$ converges, $\prod (1 - \frac{1}{2^j})$ converges to a number $\alpha$ with $0 < \alpha < 1$

Hence $m(S) = \alpha > 0$.

It remains to prove that $S$ has no interior point. Let $x$ be an arbitrary point of $S$ and let $\epsilon > 0$. Choose $m$ so that $\frac{1}{2^{n-1}} < \epsilon$. $x \epsilon S$ so $x \epsilon S_n$. $S_n = \bigcup_{i=1}^{2^n} I_{n,i}$ so for some $i$ we have $x \epsilon I_{n,i}$.

Let $y$ be the mid point of $I_{n-i,j}$, where $j = \frac{i}{2}$ or $\frac{i+1}{2}$ according as $i$ is even or odd.

$I_{n-i,j}$ is the interval giving rise to $I_{n,i}$. $y \epsilon S_{n-1}$ but $y \not\epsilon S_n$ and so $y \not\epsilon S$.

However $x \epsilon I_{n-i,j}$ so $|x - y| \leq \frac{1}{2^{n-1}} < \epsilon$

Thus $x$ is not an interior point of $S$ and so $S$ as no interior points.