Question

For an Earth satellite near the end of its life suppose

$$r^2\dot{\phi} - h_0 e^{-k\phi}$$

for some constants h_0 and k. Find the corresponding differential equation for $u(\phi)$, assuming inverse square gravity.

Answer

Newton's 2nd law: $m\{\ddot{r} - r\dot{\phi}^2\} = -\dfrac{\mu}{r^2} \quad r^2\dot{\phi} = h_0 e^{-k\phi}$

Lets put $u = \frac{1}{r}$, so $\phi = h_0 u^2 e^{-k\phi}$

$$\begin{aligned}
\dot{r} &= \dot{\phi}\frac{du}{d\phi}drdu \\
&= h_0 u^2 e^{-k\phi} - u^{-2}\frac{du}{d\phi} \\
&= -h_0 e^{-k\phi}\frac{du}{d\phi} \\
\ddot{r} &= -\dot{h}_0\left\{-ke^{-k\phi}\frac{du}{d\phi} + e^{-k\phi}\frac{d^2u}{d\phi^2}\right\}\dot{\phi} \\
&= -h_0^2 e^{-2k\phi}u^2\left\{-k\frac{du}{d\phi} + \frac{d^2u}{d\phi^2}\right\}
\end{aligned}$$

Hence the radial component of Newton's 2nd law becomes

$$\begin{aligned}
-h_0^2 e^{-2k\phi}u^2\left\{-k\frac{du}{d\phi} + \frac{d^2u}{d\phi^2}\right\} - h_0^2u^3e^{-2k\phi} &= -\frac{\mu u^2}{m} \\
\frac{d^2u}{d\phi^2} - k\frac{du}{d\phi} + u^2 &= \frac{\mu e^{2k\phi}}{mh_0^2}
\end{aligned}$$