Question

The spaceship Enterprise finds itself spiralling towards a strange planet along the path (relative to the planet)

$$r = Ae^{-k\phi}, \quad \dot{\phi} = Be^{2k\phi}.$$

(a) What force is acting on the Enterprise, in terms of r, its mass m, and the constants A and B.

(b) Find $r(t)$ for this orbit, and determine when $r = 0$ given that $\phi = 0$ when $t = 0$.

Answer

$r^2\dot{\phi} = A^2B =$ constant, therefore it is a central force as the angular momentum is constant.

$$\begin{aligned} \dot{r} &= -Ake^{-k\phi}\dot{\phi} \\ &= -kr\dot{\phi} \\ &= -kAe^{-k\phi} \cdot Be^{2k\phi} \\ &= -kABe^{k\phi} \end{aligned}$$

$$\begin{aligned} \ddot{r} &= -kABe^{k\phi}Be^{2k\phi} \\ &= -k^2AB^2e^{3k\phi} \\ &= -k^2a^4b^2r^{-3} \end{aligned}$$

$$\begin{aligned} \text{Radial compont of force} &= m(\ddot{r} - r\dot{\phi}^2) \\ &= m\left[-k^2a^2b^2r^{-3} - AB^2e^{3k\phi}\right] \\ &= -mA^4b^2(1+k^2)r^{-3} \end{aligned}$$

Therefore there is an inverse cube radial force .

Now $\dot{\phi} = Be^{2k\phi}$. Integrating gives $\int \frac{d\phi}{e^{2k\phi}} = Bt + const$

Therefore $\frac{-1}{2k}e^{-2k\phi} = Bt + C$

Now $\phi = 0$ at $t = 0 \Rightarrow c = \frac{-1}{2k} \Rightarrow Bt = \frac{1}{2k}\left(1 - e^{-2k\phi}\right)$,

whence $r = Ae^{-k\phi} = A\sqrt{1 - 2kBt}$

So $r = 0$ when $t = \frac{1}{2kB}$