Question

A space ship approaches a planet of mass M along the path (relative to the planet)

$$\frac{l}{r} = 1 + 2\cos\phi$$

(a) Show that the closest approach (if there is no collision) is to $R = \frac{1}{3}l$ at $\phi = 0$.

(b) Differentiate to find $\ddot{r}$ in terms of l and $\dot{\phi}^2$ at the point of closest approach. Use this result, and the fact that the spaceship's acceleration toward the planet is known in terms of G and M at that position, to find the speed at the closest approach in terms of G, M and l.

Answer

(a) Stationary point of $r(\phi)$ is when $r'(\phi) = 0$; $\dfrac{l}{r} = 1 + 2\cos\phi$

i.e. $\dfrac{dr}{d\phi} = -\dfrac{l}{(1+2\cos\phi)} \times -2\sin\phi = 0 \Rightarrow \phi = 0, \pi$

Inspection gives that $\phi = 0$ is the minimum value i.e. $R = \frac{l}{3}$

(b) $r = \dfrac{l}{1+2\cos\phi} \Rightarrow \dot{r} = -\dfrac{l}{(1+2\cos\phi)^2} \times -2\sin\phi\,\dot{\phi} = \dfrac{2l\sin\phi\,\dot{\phi}}{(1+2\cos\phi)^2}$

Therefore

$$\ddot{r} = 2l\left\{\frac{\cos\phi\,\dot{\phi}^2}{(1+2\cos\phi)^2} + \frac{\sin\phi\,\ddot{\phi}}{(1+2\cos\phi)^2} + \frac{4\sin\phi\,\dot{\phi}^2}{(1+2\cos\phi)^3}\right\}$$

Therefore $\ddot{r} = \dfrac{2l\dot{\phi}}{9}$ at the closest approach $\phi = 0$

Using Newton's 2nd law; radial component:

$$\begin{aligned}
m(\ddot{r} - r\dot{\phi}^2) &= -\frac{\mu}{r^2} \\
m\left(\frac{2l}{g}\dot{\phi}^2 - r\dot{\phi}^2\right) &= -\frac{\mu}{r^2} \\
m\left(\frac{2l}{9} = \frac{l}{3}\right)\dot{\phi}^2 &= -\frac{\mu}{l^2}9 \\
\dot{\phi}^2 &= \frac{81\mu}{l^3 m}
\end{aligned}$$

The speed is purely tangential at closest approach (as $\dot{r} = 0$)

so $v = r\dot{\phi} = \frac{l}{3}\sqrt{\frac{81\mu}{l^3 m}} = 3\sqrt{\frac{\mu}{ml}}$

Now $\mu = GMm$ Therefore $v = 3\sqrt{\frac{GM}{l}}$