Question

The oven in the baking room at the Glen Eyre Baked Bean factory produces molecules of a noxious gas according to a Poisson process of rate λ. Because of complaint by the workers a catalytic deodorizer has been installed which neutralizes the gas in the room at a rate of μ times the number of molecules present.

Let $p_n(t)$ denote the probability that n such molecules are present at time t. Obtain the forward differential equations for $p_n(t)$.

Assuming that a long-term equilibrium distribution exists, use the forward equations to obtain difference equations for the equilibrium probabilities. Solve these equations and deduce that the equilibrium distribution is a Poisson distribution.

The local Health and Safety Executive insists that the baking room should be totally free of this noxious gas for at least 90% of the time. What does this imply about the relationship between λ and μ?

Answer

Note: all $o(\delta t)$ terms omitted in this 'solution'.
Let $X(t)$ denote the number of molecules present at time t.

$$\begin{aligned} n=0: p_0(t+\delta t) &= P(X(t+\delta t)=0 \mid X(t)=0)p_0(t) \\ &\quad +P(X(t+\delta t)=0 \mid X(t)=1)p_1(t) \\ &= (1-\lambda\delta t)p_0(t)+(1-\lambda\delta t)\mu\delta t p_1(t) \\ &= p_0(t)-\lambda p_0(t)\delta t+\mu\delta t p_1(t) \end{aligned}$$

We deduce $p_0'(t) = -\lambda p_0(t) + \mu p_1(t)$

$$\begin{aligned} n>0: p_n(t+\delta t) &= P(X(t+\delta t)=n \mid X(t)=n)p_n(t) \\ &\quad +P(X(t+\delta t)=n \mid X(t)=n+1)p_{n+1}(t) \\ &\quad +P(X(t+\delta t)=n \mid X(t)=n-1)p_{n-1}(t) \\ &= [(1-\lambda\delta t)(1-n\mu\delta t)+\lambda\delta t\cdot n\mu\delta t]p_n(t) \\ &\quad +[(1-\lambda\delta t)(n+1)\mu\delta t]p_{n+1}(t) \\ &\quad +[\lambda\delta t(1-(n-1)\mu\delta t]p_{n-1}(t) \\ &= (1-(\lambda+n\mu)\delta t)p_n(t)+(n+1)\mu\delta t p_{n+1}t \\ &\quad +\lambda\delta t p_{n-1}(t) \end{aligned}$$

We deduce $P_n'(t) = -(\lambda +_n \mu)p_n(t) + (n+1)\mu p_{n+1}(t) + \lambda p_{n-1}(t)$.
Assuming that $p_n(t) to \pi_n$ as $t \to \infty$ then $p_n'(t) \to 0$ as $t \to \infty$ for all n.
We obtain

$0 = -\lambda\pi_0 + \mu\pi_1$ i.e. $\mu\pi_1 = \lambda\pi_0$

$0 = -(\lambda + n\, mu)\pi_n + (n+1)\mu\pi_{n+1} + \lambda\pi_{n-1}$

$$\begin{aligned}
(n+1)\mu\pi_{n+1} &= (\lambda+n\mu)\pi_n - \lambda\pi_{n-1} \\
n=1: 2\mu\pi_2 &= (\lambda+\mu)\pi_1 - \lambda\pi_0 = \lambda\pi_1 \\
n=2: 3\mu\pi_3 &= (\lambda+2\mu)\pi_2 - \lambda\pi_1 = \lambda\pi_2
\end{aligned}$$

By induction we can then establish

$$n\mu\pi_n = \lambda\pi_{n-1} \text{ i.e. } \pi_{\mathrm{n}} = \frac{1}{\mathrm{n}} \cdot \frac{\lambda}{\mu}\pi_{\mathrm{n-1}}$$

This then gives $\pi_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n \pi_0$.

For this to be a probability distribution $\sum_{n=0}^{\infty} \pi_n = 1$.

so $\pi_0 \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n = 1$ i.e. $\pi_0 = e^{-\frac{\lambda}{\mu}}$

and $\pi_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n e^{-\frac{\lambda}{\mu}}$

so the equilibrium distribution is Poisson with parameter $\frac{\lambda}{\mu}$.

The proportion of time for which gases are not emitted is $\pi_0 = e^{-\frac{\lambda}{\mu}}$.

So we must have $e^{-\frac{\lambda}{\mu}} \geq \frac{9}{10}$.

$$\text{i.e. } \lambda \leq \mu \ln \frac{10}{9} (\mu \leq 9.49 \cdots \times \lambda)$$