Question

Explain what a branching Markov chain is, and state clearly the fundamental theorem for branching chains. Suppose we begin with a single individual. Let $F_n(s)$ denote the probability generating function for the number of individuals in generation n, and let $A(s)$ denote the probability generating function for the number of offspring of any individual. Let P denote the probability of eventual extinction. Explain why

$$P + \lim_{n\to\infty} F_n(0)$$

Assuming the relationship $F_n(s) = A(F_{n-1}(s))$, prove the fundamental theorem for branching chains.

The exquisite Lewis orchid reproduces as follows. Each plant produces 0, 1, 2 or 3 seeds, with equal probability. Then when Autumn comes the plant dies. In the Spring a seed may be dead, or it may produce a new plant, with equal probabilities. Each plant and each seed behaves independently of all the others.

Find the probability of eventual extinction of this species of orchid.

Answer

Suppose we have a population of individuals each reproducing independently of the others, and that the probability distributions for the number of offspring of all individuals in generation n. Then $X(n)$ is called a branching Markov chain.

The Fundamental Theorem states:

(i) The probabilities of extinction, when the population has size 1 initially, is the smallest positive root of the equation $x = A(x)$, where A is the p.g.f. of the number of offspring per individual.

(ii) Extinction occurs with probability 1 if and only if $\mu \leq 1$, where μ is the mean number of offspring per individual.

The probabilities of extinction at or before the nth generation if
$P(X_n = 0) = F_n(0)$.
This is a non-decreasing function of n. The probability of eventual extinction is the union of the increasing sequence of events $(X_n = 0)$. So $P = \lim_{n\to\infty} F_n(0)$.
Since $F_n(0) = A(F_{n-1}(0))$, taking limits gives

$$P = A(P)$$

so P is a root of $x = A(x)$, satisfying $0 \leq x \leq 1$.

Let $A(x) = a_0 + a_1x + a_2x + \cdots$

If $a_0 = 0$ then each individual has at least one offspring, and extinction is impossible. $\mu \geq 1$ in this case. If $a_0+a_1 = 1$ then each individual has at most one offspring. If $a_1 < 1$ then the probability that the population survives to generation n is $(a_1)^n \to 0$ as $nto\infty$, so extinction is effectively certain.

Now $A'(x) = a_1 + 2a_2x + 3a_3a^2 + \cdots > 0$ for $x > 0$, for otherwise all the a's are zero, giving $a_0 = 1$ and extinction at the first generation.

$A''(x) = 2a_2 + 6a_3x + \cdots > 0$ for all x, for otherwise $a_0 + a_1 = 1$, which we have already dealt with. So $A(x)$ is convex and cuts the line $y = x$ in at most 2 places for $x > 0$. Now $A(1) = 1$, so one root is $x = 1$.

Let P be the smaller of the two roots. We show by induction that $F_n(0) < P$ for all n.

$F_0(0) = 0 < P$ since $F_0(s) = s$.

Suppose then that $F_n(0) < P$.

Then $F_{n+1}(0) = A(F_n(0)) < A(P) = P$, since A is strictly increasing.

So $\lim_{n\to\infty} F_n(0)$ must be the smaller of the two roots.

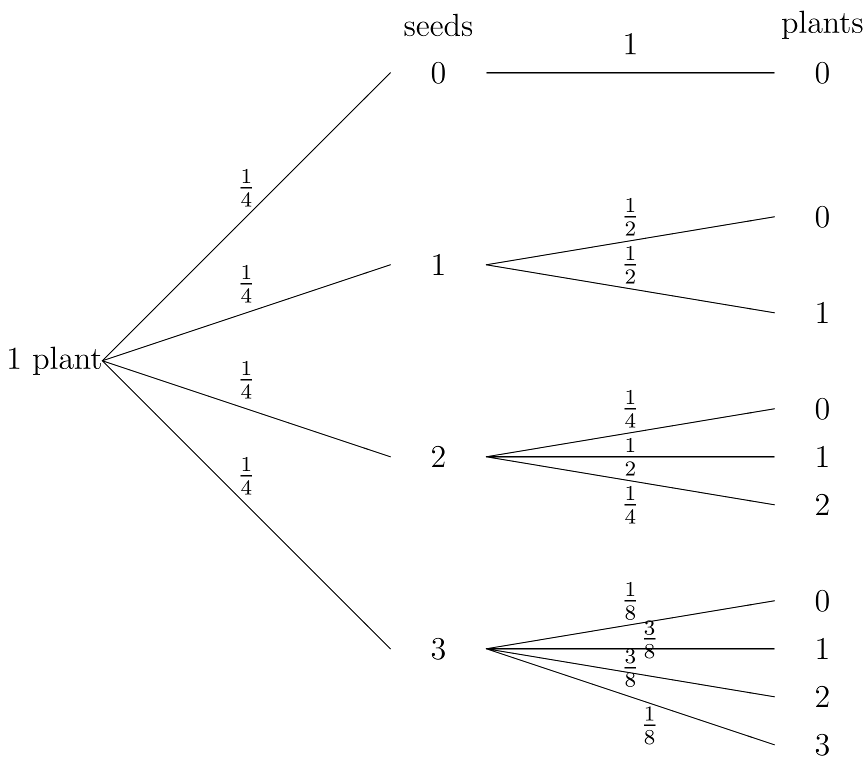
We consider 3 cases. Let $A(x) = x$ have roots $P_1,\ P_2$

1. $P_1 < P_2 = 1$ PICTURE

2. $1 = P_1 < P_2$ PICTURE

3. $P_1 = P_2 = 1$ PICTURE

Now $\mu = A'(1)$. In case 1 $\mu > 1$. In case 2 $\mu < 1$ and in case 3 $\mu = 1$. Extinction happens with probability 1 in cases 2 and 3, i.e. when $\mu \leq 1$.



So the probability distribution for the number of plants produced from one plant is:

$p_0 : \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{8} = \frac{15}{32}$

$p_1 : \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{3}{8} = \frac{11}{32}$

$p_2 : \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{3}{8} = \frac{5}{32}$

$p_3 : \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$

EITHER

$A(x) = \frac{1}{32}(15 + 11x + 5x^2 + x^3)$

$x = A(x)$ gives $x^3 + 5x^2 - 21x + 15 = 0$

i.e. $(x-1)(x^2 + 6x - 15) = 0$

The roots are $x = 1,\ -3 \pm 2\sqrt{6}$. Now $-3 + 2\sqrt{6} > 1$ so $P = 1$.

OR

The mean number of offspring per individual is

$$1 \times \frac{11}{32} + \frac{2\ times 5}{32} + \frac{3 \times 1}{32} = \frac{24}{32} < 1$$

so again $P = 1$.