Question

A signalling lamp has three possible modes. off, red, and green. A secret agent has two such lamps. He operates them at one minute intervals according to the following scheme. He selects a lamp at random. If it is off he switches it to red or green with equal probabilities. If it is already on, he changes the colour being shown. The two lamps are indistinguishable. Write down the six possible states of the system and explain briefly why it forms a Markov chain. Classify the states according to transience or recurrence, giving reasons in each case. Determine the period of any recurrent states. Calculate the mean recurrence time of any positive recurrent states.

What proportions of a long interval of time would you expect the system to spend in each state?

The system starts with both lamps off, and the agent commences one minute later. What is the probability that one lamp is off after $\left(k+\frac{1}{2}\right)$ minutes have elapsed?

Answer

The six possible states are as follows:
1. Both lamps off 2. 1 off 1 red 3. 1 off 1 green
4. Both red 5. Both green 6. 1 red 1 green

When the agent arrives to change the lamps, the probabilities of change depend only on the state the lamps are in, and not on how that state was reached. This is the characteristic property of a Markov chain. The transition matrix is as follows, the rows/columns corresponding to states 1-6 above.

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

The transition diagram can be drawn as follows
PICTURE

State 1 is transient. Indeed the system leaves state 1 after 1 step, never to return. $p_{11} = 0$.

States 2 and 3 are transient. $p_{22} = p_{33} = \frac{1}{4}$. Also from states 2 and 3 there is a probability $\frac{1}{2}$ if entering the closed set of states $\{4,\ 5,\ 6\}$.

The closed set $\{4\,5,\ 6\}$ forms a sub-chain. It is finite and irreducible - i.e. all states intercommunicate. Thus they are all positive recurrent. Clearly they all have period 2.

Now $p_{66}^{(}2) = 1$ and clearly $\mu_6 = 2$.
For state 4, paths of return are as follows:
$4{-}6{-}4 \left(\text{prob } \frac{1}{2}\right)$ $4{-}6{-}5{-}6{-}4 \left(\text{prob } \frac{1}{4}\right)$ $4{-}6{-}5{-}6{-}5{-}6{-}4 \left(\text{prob } \frac{1}{8}\right)$
etc.
so

$$\begin{aligned}
\mu_4 &= 2 \times \frac{1}{2} + 4 \times \frac{1}{4} + 6 \times \frac{1}{8} + 8 \times \frac{1}{16} + \cdots + 2n \times \frac{1}{2}n + \cdots \\
2\mu_4 &= 2 + 4 \times \frac{1}{2} + 6 \times \frac{1}{4} + 8 \times \frac{1}{8} + 10 \times \frac{1}{16} + \cdots \\
&\quad + (2n+2) \times \frac{1}{2}n + \cdots \\
\mu_4 &= 2 + 2 \times \frac{1}{2} + 2 \times \frac{1}{4} + 2 \times \frac{1}{8} + 2 \times \frac{1}{16} + \cdots = 4
\end{aligned}$$

Also $\mu_5 = 4$ by symmetry.

Occupancy proportions are the reciprocal of mean recurrence times so for state 6 $\frac{1}{2}$ the time and for states 4 and 5 $\frac{1}{4}$ each.
After one minute (i.e. one operation) 1 lamp is off. If it is to remain off then the other lamp must be chosen on each of the $k-1$ subsequent occasions i.e. probability $\frac{1}{2^{k-1}}$.