Question

A simple random random walk has the set $\{0,\ 1,\ 2,\ cdots,\ a-1,\ 1\}$ as possible states. States 0 and a are reflecting barriers from which reflection is certain, i.e., if the random walk is in state 0 or a at step n the it will be in state 1 or state $a-1$ respectively at step $n+1$. For all other states, transitions of $+1,\ -1,\ 0$ take place with non-zero probabilities $p,\ q,\ 1-p-q$ respectively.

Let $p_{j,\ k}^{(n)}$ denote the probability that the random walk is in state k at step n, having started in state j. Obtain difference equations relating these probabilities, for the cases $k=0,\ 1,\ a,\ a-1$ and $1<k<a-1$.

Assuming that there is a long-term equilibrium probability distribution (π_k), where

$$\pi_k = \lim_{n\to\infty} p_{j,\ k}^{(n)} \text{ for } 0 \le \text{j} \le \text{a},$$

use the difference equations derived to obtain difference equations of π_k. Solve these equations recursively, or other-wise, to obtain explicit formulae for π_k in terms of $p,\ q$ and a.

Answer

$$\begin{aligned}
p_{j,\ 0}^{(n)} &= q\cdot p_{j,\ 1}^{(n-1)}\\
p_{j,\ 1}^{(n)} &= p_{j,\ 0}^{(n-1)} + q\cdot p_{j,\ 2}^{(n-1)} + (1-p-q)p_{j,\ 1}^{(n-1)}\\
p_{j,\ k}^{(n)} &= p\cdot p_{j,\ k-1}^{(n-1)} + q\cdot p_{j,\ k+1}^{(n-1)} + (1-p-q)p_{j,\ k}^{(n-1)} \quad 1<k<a-1\\
p_{j,\ a-1}^{(n)} &= p\cdot p_{j,\ a-2}^{(n-1)} + p_{j,\ a}^{(n-1)} + (1-p-q)p_{j,\ a-1}^{(n-1)}\\
p_{j,\ a}^{(n)} &= p\cdot p_{j,\ a}^{(n-1)}
\end{aligned}$$

Assuming the existence of an equilibrium distribution, taking limits in the above equations gives:

$$\begin{aligned}
\pi_0 &= q\pi_1 && (1)\\
\pi_1 &= \pi_0 + q\pi_2 + (1-p-q)\pi_1 && (2)\\
p_k &= p\pi_{k-1} + q\pi_{k+1} + (1-p-q)\pi_k && (3)\\
\pi_{a-1} &= p\pi_{a-2} + \pi_a + (1-p-q)\pi_{a-1} && (4)\\
\pi_a &= p\pi_{a-1} && (5)
\end{aligned}$$

Equation (1) gives $\pi_1 = \frac{1}{q}\pi_0$.

(2) gives

$$\pi_2 = \frac{(p+q)}{q}\pi_1 - \frac{1}{q}\pi_0 = \frac{(p+q)}{q}\cdot\frac{1}{q}\pi_0 - \frac{1}{q}\pi_0 = \frac{1}{q}\left(\frac{p}{q}\right)\pi_0$$

(3) gives

$$\pi_3 = \frac{(p+q)}{q}\pi_2 - \frac{p}{q}\pi_1 = \frac{(p+q)}{q}\cdot\frac{1}{q}\cdot\frac{p}{q}\pi_0 - \frac{p}{q}\cdot\frac{1}{q}\pi_0 = \frac{1}{q}\left(\frac{p}{q}\right)^2\pi_0$$

Assume $\pi_i = \frac{1}{q}\left(\frac{p}{q}\right)^{i-1}$ for $1 \leq i \leq k$.

(3) gives

$$\pi_{k+1} = \frac{p+q}{p}\pi_k - \frac{p}{q}\pi_{k-1} = \frac{p+q}{q}\cdot\frac{1}{q}\left(\frac{p}{q}\right)^{i-1}\pi_0 - \frac{p}{q}\frac{1}{q}\left(\frac{p}{q}\right)^{i-2}\pi_0 = \frac{1}{q}\left(\frac{p}{q}\right)^{i}\pi_0$$

so $\pi_k = \frac{1}{q}\left(\frac{p}{q}\right)^{k-1}\pi_0$ for $1 \leq k \leq a-1$.

(5) then gives $\pi_a = p\pi_{a-1} = \left(\frac{p}{q}\right)^{a-1}\pi_0$

(This is consistent with (4) as expected)

Now $\sum_{k=0}^{a}\pi_k = 1$ and so

$$\pi_0\left(1 + \frac{1}{q}\left(1 + \frac{p}{q} + \cdots + \left(\frac{p}{q}\right)^{a-2}\right) + \left(\frac{p}{q}\right)^{a-1}\right) = 1$$

i.e. $\pi_0 = \left(1 + \frac{1-(\frac{p}{q})^{a-1}}{q-p} + \left(\frac{p}{q}\right)^{a-1}\right)^{-1}$

$p \neq q$

which gives the formulae3 for π_k, as these are in terms of π_0 above.