Question
A gambler with initial capital z plays against an opponent with initial capital $(a-z)$, where a and z are integers with $0`z \leq a$. At each play the gambler wins 1 with probability p and loses 1 wit probability q. A draw occurs with probability $r < 1$, where $p+q+r-1$, and in the case of a draw both players retain their 1 stake.

Let q_z denote the probability that the gambler will eventually be ruined. Write down a difference equation for q_z and solve it to obtain an explicit formula for q_z in the case $p \neq q$.

The two players initially have 10 between them. If $p = 0.4$ and $q = 0.3$ how much of the 10 must the gambler have in order to have a better than even chance of not being ruined?

With these values of p and q, how much initial capital does the gambler need in order to have a better than even chance of not being ruined when playing against an infinitely rich opponent?

Answer

$$\begin{aligned}
\text{P(ruin)} &= \text{P(ruin and win 1st bet)} \\
&\quad + \text{P(ruin and lose 1st bet)} \\
&\quad + \text{P(ruin and draw 1st bet)} \\
&= \text{P(ruin — wins 1st bet)P(win 1st bet)} \\
&\quad + \text{P(ruin — loses 1st bet)P(lose 1st bet)} \\
&\quad + \text{P(ruin — draws 1st bet)P(draw 1st bet)}
\end{aligned}$$

so

$$\begin{aligned}
q_z &= q_{z+1} \cdot p + q_{z-1} \cdot q + q_z \cdot r \\
q_z(1-r) &= p \cdot q_{z+1} + q \cdot q_{z-1} \\
q_z &= \frac{p}{p+q} q_{z+1} + \frac{q}{p+q} q_{z-1} \quad (r \neq 1)
\end{aligned}$$

with boundary conditions $q_0 = 1;\ q_a = 0$
Substituting $q_z = \lambda^z$ gives

$$\lambda^z = \frac{p}{p+q}\lambda^{z+1} + \frac{q}{p+q}\lambda^{z-1}$$

$$\begin{aligned}
\text{i.e. } \frac{\mathrm{p}}{\mathrm{p}+\mathrm{q}}\lambda^2 - \lambda + \frac{\mathrm{q}}{\mathrm{p}+\mathrm{q}} &= 0 \\
(\lambda - 1)\left(\frac{p}{p+q}\lambda - \frac{q}{p+q}\right) &= 0
\end{aligned}$$

so $\lambda = 1$ or $\lambda = \dfrac{p}{q} \neq 1$ when $p \neq q$.

So the general solution is

$$q_z = A + B\left(\frac{q}{p}\right)^z$$

The boundary conditions give

$$\begin{aligned}
1 &= A + B \\
0 &= A + B\left(\frac{q}{p}\right)^a
\end{aligned}$$

so $1 = B\left(1 - \left(\dfrac{q}{p}\right)^a\right)$

Thus $B = \dfrac{1}{1-(\frac{q}{p})^a}$ and $A = 1 - B = \dfrac{-(\frac{q}{p})^a}{1-(\frac{q}{p})^a}$

Hence $q_z = \dfrac{(\frac{q}{p})^z - (\frac{q}{p})^a}{1-(\frac{q}{p})^a}$

Now with $p = 0.4,\ q = 0.3$, and $a = 10$ we have

$$q_z = \frac{(\frac{3}{4})^z - (\frac{3}{4})^{10}}{1-(\frac{3}{4})^{10}}$$

so $q_z < \dfrac{1}{2}$ if and only if $\left(\dfrac{3}{4}\right)^z - \left(\dfrac{3}{4}\right)^{10} < \dfrac{1}{z}\left(1 - \left(\dfrac{3}{4}\right)^{10}\right)$

i.e. $2\left(\dfrac{3}{4}\right)^z < 1 + \left(\dfrac{3}{4}\right)^{10}$

$$z > \frac{\ln\left(1 + \left(\frac{3}{4}\right)^{10}\right) - \ln 2}{\ln\left(\frac{3}{4}\right)} = 2.219\cdots$$

Playing against an infinitely rich opponent is analysed by letting $a \to \infty$, giving the probability of ruin as $\left(\dfrac{3}{4}\right)^z$.

Now $\left(\dfrac{3}{4}\right)^z < \dfrac{1}{2}$ if and only if $z > \dfrac{\ln(\frac{1}{2})}{\ln(\frac{3}{4})} = 2.41\cdots$

so in fact with 3 the gambler has a better than even chance of not being ruined against an infinitely rich opponent.