Exam Question

Topic: SurfaceIntegral The dean's trophy presented to the best lecturer of the year is in the shape of the upper half of a cylinder, specified by

$$y^2 + z^2 = 1,\ z \geq 0,\ 0 \leq x \leq 1.$$

The curved part of the surface is to be covered with an iridescent foil made of a mixture of precious metals, where the composition varies so as to achieve a colour change over the surface. The density at a point (x, y, z) on the surface is given by $(1 + 2x^2)$.

Calculate the total mass of foil by evaluating an appropriate surface integral.

Solution

The equation of the surface can be rewritten as $z = \sqrt{1 - y^2}$, so

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{1 - y^2}};\ \frac{\partial z}{\partial x} = 0.$$

Therefore we have

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{y^2}{1 - y^2}} = \frac{1}{\sqrt{1 - y^2}}$$

The mass is therefore given by

$$\begin{aligned} M &= \int_0^1 dx \int_{-1}^1 \frac{1 + 2x^2}{\sqrt{1 - y^2}}\, dy = \int_0^1 (1 + 2x^2)\, dx \int_{-1}^1 \frac{1}{\sqrt{1 - y^2}}\, dy \\ &= \left[x + \frac{2}{3}x^3\right]_0^1 \left[\sin^{-1} y\right]_{-1}^1 = \frac{5\pi}{3}. \end{aligned}$$