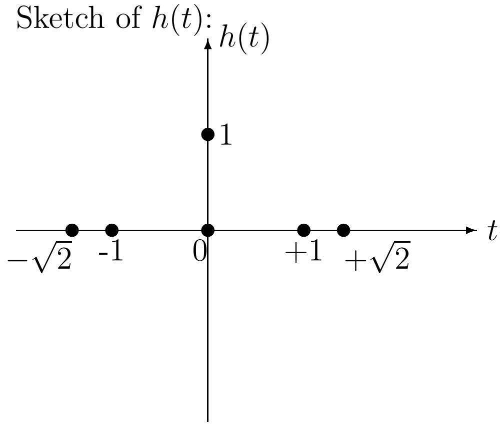
Question
Use Laplace's method to show that

$$\int_{-1}^{1} dt,\ e^{-x(2t^2-t^4)}\sqrt{2+t} \sim \sqrt{\frac{\pi}{x}},\ x \to +\infty$$

and obtain the corresponding result for

$$\int_{0}^{3} dt,\ e^{-x(2t^2-t^4)}\sqrt{2+t} \quad x \to +\infty$$

Answer
$I = \int_{-1}^{+1} dt\, e^{-x(2t^2-t^4)}\sqrt{2+t}$
$h(t) = 2t^2 - t^4,\ h'(t) = 4t - 4t^3, \Rightarrow t = 0$ or ± 1 are min/max
$h''(t) = 4 - 12t^2 \Rightarrow \begin{array}{l} t = 0 : h''(0) = 4 > 0 \Rightarrow min \\ t = \pm 1 : h''(\pm 1) = -8 < 0 \Rightarrow max \end{array}$
Don't contribute at leading order of exponentials
Sketch of $h(t)$:



Thus we have a full contribution from the min. at $t=0$.
Set

$$\begin{aligned} u^2 &= h(t) - \overbrace{h(0)}^{=0} = 2t^2 - 2^4 \\ 2u\,du &= h'(t)\,dt \end{aligned}$$

Therefore $I = \int_{-\sqrt{h(-1)-h(0)}}^{\sqrt{h(1)-h(0)}} e^{-xu^2 - xh(0)} \underbrace{\frac{\sqrt{2+t(u)}}{h'(t(u))}}_{\text{Need to expand this about } t=0\ (u=0)} \cdot 2u,\ du$

$h'(t) approx h''(0)t \quad t \to +\infty$ by Taylor

and $u^2 = h(t) - h(0) \approx \dfrac{h''(0)t^2}{2},\ t \to +\infty$

$\Rightarrow u = \pm\sqrt{\dfrac{h''(0)}{2}}t$ (take $+\sqrt{}$ as t increases when u increases.)

Therefore

$$\begin{aligned} I &\approx \int_{-\sqrt{h(1)-h(0)}}^{\sqrt{h(1)-h(0)}} e^{-xu^2}\ 2u\ \frac{\sqrt{2+t(u)}}{h''(0)u\sqrt{\frac{2}{h''(0)}}}\,du \\ &\approx 2\int_{-\infty}^{+\infty} \frac{e^{-xu^2}}{\sqrt{h''(0)}}\,du \quad x \to +\infty \\ &\quad \text{errors are exponentially small as } x \to +\infty \\ &= \frac{2}{\sqrt{4}}\int_{-\infty}^{+\infty} e^{-xu^2}\,du \\ &= \sqrt{\frac{\pi}{x}} \quad x \to +\infty \end{aligned}$$

(By standard Gaussian integral)

KNOW: $\int_{-\infty}^{+\infty} e^{-\alpha x^2}\,dx = \sqrt{\dfrac{\pi}{\alpha}}$

When we consider

$$\int_0^3 dt\, e^{-x(2t^2 - t^4)}\sqrt{2+t},\ x \to +\infty$$

Looking at the graph of $h(t) = 2t^2 - t^4$ as above, we see that $t=0$ is an endpoint minimum but $t=3$ is an overall minimum on the range of integration. Thus the dominant behaviour will come from $t=3$, a linear endpoint. Thus we proceed as in question 6:

$$u = h(t) - \underbrace{h(3)}_{} = h(t) + 63$$

$$2 \times 9 - 81$$

$$du = h'(t)\,dt$$

NB endpoint is at RHS

$$J = \int_0^3 e^{-x(2t^2-t^4)}\sqrt{2+t}\,dt \sim \int_{\substack{h(0)-h(3)=+63 \\ \text{NB}}}^{0} e^{-xu+63x}\frac{\sqrt{2+t(u)}}{h'(t(u))}\,du$$

$$\begin{aligned} u = h(t) - h(3) &\approx h'(3)(t-3)t \to 3 \\ &= (4\cdot 3 - 4\cdot 3^3)(t-3) \\ &= -96(t-3) \end{aligned}$$

NB minus sign will reverse limits in integral

$$\begin{aligned} h'(t) &\approx h'(3) + h''(3)(t-3) \text{ by Taylor} \\ &= -96 \text{ toleadingorder} \end{aligned}$$

Also need expansion of $\sqrt{2+t}$ about $t = +3$:
$\sqrt{2+t} = \sqrt{5} + O(t-3)$ $t \to 3$ byTaylor
Therefore

$$\begin{aligned} J &\sim \int_{63}^{0} e^{-xu+63x}\frac{\sqrt{5}}{-96}\,du \quad x \to +\infty \\ &= +e^{63x}\frac{\sqrt{5}}{96}\int_0^{63} e^{-xu}\,du \; (-\int_{63}^{0} = +\int_0^{63}) \\ &\sim e^{63x}\frac{\sqrt{5}}{96}\int_0^{\infty} e^{-xu}\,du \quad x \to +\infty \\ &\sim e^{63x}\frac{\sqrt{5}}{96x} \quad x \to +\infty \end{aligned}$$

The two integrals above only differ by their range of integration. However they have dramatically different behaviour as $x \to +\infty$. The first tends to zero $\left(\frac{1}{\sqrt{x}} \to 0 \text{ as x} \to +\infty\right)$, the second $\to +\infty$ ($e^{63x} \to \infty$ as x $\to +\infty$). In fact, even when $x = +1$, $e^{63} \approx 2.3 \times 10^{27}$.