Question

Show that if $a > 0,\ n > 0,\ f(0) \neq 0$

$$\int_0^a e^{-xt^n} f(t)\,dt \sim \frac{\Gamma(\frac{1}{n})}{nx^{\frac{1}{n}}} f(0),\ x \to +\infty$$

Answer

$$I(x) = \int_0^a e^{ixt^n} f(t)\,dt.\quad \underbrace{a > 0}_{\substack{\text{Integral along}\\ \text{real line.}}},\ \overbrace{n > 0}^{\text{No singularity at } t=0.},\ \underbrace{f(0) \neq 0}_{\substack{\text{No vanishing}\\ \text{at endpoint.}}}$$

This has an endpoint minimum at $t = 0$:

$h(t) = t^n \Rightarrow h'(0) = nt^{n-1}|_{t=0} = 0$

Indeed $h'(0) = h''(0) = h'''(0) = \cdots = h^{(n-1)}(0) = 0$

$$h^{(n)}(0) = n!$$

$$\text{Thus set}\quad \begin{aligned} u &= h(t) - h(0) = t^n \\ du &= nt^{n-1}\,dt \text{ (exactly)} \\ &= nu^{\frac{n-1}{n}}\,dt \text{ (exactly)} \end{aligned}$$

Therefore $I(x) = \int_0^{a_n} e^{-xu} \frac{f(u^{\frac{1}{n}})}{nu^{\frac{n-1}{n}}}\, du$
Now for fixed finite n the upper limit can be put to infinity with only exponentially small error. (NB. ???? if $a > 1$)

$$
\begin{aligned}
I(x) &\approx \int_0^{\infty} e^{-xu} \frac{f(u^{\frac{1}{n}})}{n(u^{\frac{n-1}{n}})}\, du \quad (\text{limit} \to +\infty) \quad \text{x} \to +\infty \\
&\approx \int_0^{\infty} \frac{e^{-xu}}{u^{1-\frac{1}{n}}}\, du \frac{f(0)}{n} (\text{expand about u} = 0 \quad \text{x} \to +\infty \\
&\sim \frac{\Gamma(\frac{1}{n} - 1 + 1)}{nx^{\frac{1}{n}-1+1}} f(0) \\
&\sim \frac{\Gamma(\frac{1}{n}) f(0)}{nx^{\frac{1}{n}}} \quad as x \to +\infty
\end{aligned}
$$

e.g., $n = 2 \sim \sqrt{\pi}\frac{f(0)}{2\sqrt{x}}$ in agreement with lecture notes.