**Question**

Given that $b > a,\ \lambda > -1,\ h(x) > h(a),\ h'(a) \neq 0$, show that as $x \to +\infty$

$$I(x) = \int_a^b (t-a)^\lambda e^{-xh(t)}\,dt \sim \frac{e^{-xh(a)}\Gamma(\lambda+1)}{[xh'(a)]^{\lambda+1}}.$$

**Answer**

$I(x) = \displaystyle\int_a^b (t-a)^\lambda e^{-x\ln(t)}\,dt$

For convergence at $t = a$. It's a linear endpoint.

$\underbrace{b > a}, \ \overbrace{\lambda > -1}, \ \underbrace{h(x) > h(a)}, \ \overbrace{h'(a) \neq 0}$

Integral on real line.

Endpoint at $x = a$ will dominate for $x > a$.

This is the general case of question 6: a dominant linear endpoint as $x \to +\infty$ where $f(t)$ vanishes there.

Proceed as above:

$$\begin{aligned} u &= h(t) - h(a) \\ &= h'(a)(t-a) + O[(t-a)^2] \end{aligned}$$

$\Rightarrow u \approx h'(a)(t-a) \quad (1)$

Therefore $I(x) = e^{-x\ln(a)} \displaystyle\int_0^{h(b)-h(a)} e^{-xu}\frac{[t(u)-a]^\lambda}{h'(t(u))}\,du$

$(1) \Rightarrow t - a \approx \dfrac{u}{h'(a)}$

Therefore to leading order we have: $h'(t(u)) = h'(a)$

$I(x) \approx e^{-x\ln(a)} \displaystyle\int_0^{h(b)-h(a)} \frac{e^{-xu}u^\lambda}{[h'(a)]^{\lambda+1}}\,du$

as $x \to +\infty$ for Poincaré the upper limit can be set to $\infty$ (with exponentially small error).

Hence

$$
\begin{aligned}
I(x) &\sim \frac{e^{-x\ln(a)}}{[h'(a)]^{\lambda+1}} \int_0^\infty e^{-xu}\ u^\lambda\, du \\
&\quad \text{this is a known integral} \\
&\sim \frac{e^{-x\ln(a)}\Gamma(\lambda+1)}{[xh'(a)]^{\lambda+1}}
\end{aligned}
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