Question

Use Watson's lemma to calculate the full Poincaré asymptotic expansion of

(a) $\int_0^\infty e^{-xt}\log(1+\sqrt{t})\,dt,\ x\to+\infty$

(b) $\int_0^1 \frac{e^{-xt}}{\sqrt{t(2+t)}}\,dt,\ x\to+\infty$

(c) $\int_0^\infty \frac{e^{-xt}}{\sqrt{t(2+t)}}\,dt,\ x\to+\infty$

Answer

With Watson's lemma we must check existence of expansion of $f(t)$ about $t=0$ and $|f(t)|<Ae^{bt}$.

$\int_0^\infty e^{-xt}f(t)\,dt$ for some finite $A,\ b$.

(a) $\int_0^\infty e^{-xt}\log(1+\sqrt{t})$

Clearly $|log(1+\sqrt{t})|<e^t,\quad t>0$

Also $\log(1+\sqrt{t})=\sum_{n=0}^\infty \frac{t^{\frac{n}{2}}}{n}(-1)^n$ by Maclaurin / Taylor series about $t=0$.

Thus $\int_0^\infty e^{-xt}\log(1+\sqrt{t})\,dt\sim\sum_{n=0}^\infty\frac{(-1)^n}{n}\int_0^\infty e^{-xt}t^{\frac{n}{2}}$.

$\left(\lambda_n=\frac{n}{2},\ a_n=\frac{(-1)^n}{n}\right)=\sum_{n=0}^\infty\frac{(-1)^n}{n}\frac{\Gamma(\frac{n}{2}+1)}{x^{\frac{n}{2}}+1},\quad x\to+\infty$ by Watson.

(b) $\displaystyle\int_0^\infty \frac{e^{-xt}}{\sqrt{t(2+t)}}\,dt = \left[\int_0^1 + \int_1^\infty\right] \frac{e^{-xt}}{\sqrt{t(2+t)}}\,dt$

As framed in notes, we only have Watson for $\displaystyle\int_0^\infty$-type integrals.

BUT $\displaystyle\int_1^\infty$ is exponentially small as $x \to +\infty$:

$$
\begin{aligned}
\left|\int_1^\infty \frac{e^{-xt}}{\sqrt{t(2+t)}}\,dt\right| &\leq \int_1^\infty \frac{e^{-xt}}{\sqrt{t(2+t)}}\,dt \\
&\quad \sqrt{t(2+t)} \geq \sqrt{3} \text{ for all t} \geq 1 \\
&\leq \int_1^\infty \frac{e^{-xt}}{\sqrt{3}}\,dt \\
&= \frac{e^{-x}}{x\sqrt{3}} \\
&= O\left(\frac{e^{-x}}{x}\right)
\end{aligned}
$$

So for Poincaré purposes,

$$\int_0^\infty dt \sim \int_0^1 dt$$

Thus we apply Watson. Must check:

(i) $f(t) = \dfrac{1}{\sqrt{t(2+t)}} < Ae^{b-t}$ for all $t > 0$, for some $A,\ b > 0$

This is clearly satisfied by $A = 1,\ b = 1$, i.e.,

$$\frac{1}{\sqrt{t(2+t)}} < e^t \quad \surd$$

(ii) $f(t)$ has an expansion about $t = 0$.

$f(t) = \dfrac{1}{\sqrt{2t}}\left(1+\dfrac{t}{2}\right)^{-\frac{1}{2}} = \dfrac{1}{\sqrt{2t}}\displaystyle\sum_{s=0}^{\infty}\left(\frac{t}{2}\right)^s \frac{(-1)^s}{s!}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(\frac{1}{2})} \quad \surd$

We can work this last fraction out by using factorial notation to spot this as $\dfrac{(s-\frac{1}{2})!}{(-\frac{1}{2})!}$ and converting back to Γ functions by adding one to top and bottom.

Thus $\lambda_s - \dfrac{1}{2}$

$d_s = \dfrac{(-1)^s}{2^{s+\frac{1}{2}}} \dfrac{\Gamma(s+\frac{1}{2})}{s!\Gamma(\frac{1}{2})}$

Therefore, from Watson:

$$\int_0^1 \frac{e^{-xt}}{\sqrt{t(2+t)}}\, dt \sim \sum_{s=0}^{\infty} \frac{(-1)^s \Gamma(s+\frac{1}{2})}{2^{s+\frac{1}{2}} s! \Gamma(\frac{1}{2})} \frac{\Gamma(s+\frac{1}{2})}{x^{s+\frac{1}{2}}} \quad x \to +\infty$$

Now $\Gamma(\frac{1}{2}) = \sqrt{\pi}$: KNOW THIS FOR THE EXAM!

Therefore $\sim \sum_{s=0}^{\infty} \dfrac{(-1)^s [\Gamma(s+\frac{1}{2})]^2}{2^{s+\frac{1}{2}} s! \sqrt{\pi} x^{s+\frac{1}{2}}}, \quad x \to +\infty$

(c) From above, for Poincaré expansion

$$\int_0^1 \sim \int_0^{\infty}$$

so $\displaystyle\int_0^{\infty} \frac{e^{-xt}}{\sqrt{t(2+t)}} \sim \sum_{s=0}^{\infty} \frac{(-1)^s [\Gamma(1+\frac{1}{2})]^2}{2^{s+\frac{1}{2}} s! \sqrt{\pi} x^{s+\frac{1}{2}}}$