Question
Show that as $x \to \infty$

$$\int_x^\infty \frac{\cos t}{\sqrt{t}}\,dt \sim \frac{1}{\sqrt{x}}(f\cos x - g\sin x)$$

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where

$$f \sim \frac{1}{2x} - \frac{1.3.5}{(2x)^3} + \frac{1.3.5.7.9}{(2x)^5}$$

$$g \sim 1 - \frac{1.3}{(2x)^2} + \frac{1.3.5.7}{(2x)^4}$$

Comment on the asymptotic validity of the results.

Answer
Consider $J = \int_x^\infty \frac{e^{it}}{\sqrt{t}}\,dt = \int_x^\infty \frac{\cos t}{\sqrt{t}}\,dt + i\int_x^\infty \frac{\sin t}{\sqrt{t}}\,dt$
so can do both integrals by just looking at J.
Use integration by parts

$$\begin{aligned} u &= \frac{1}{i\sqrt{t}} & dv &= ie^{it} \\ du &= -\frac{1}{2i}t^{-\frac{3}{2}} & v &= e^{it} \end{aligned}$$

$$J = \left[\frac{e^{it}}{i\sqrt{t}}\right]_x^\infty + \frac{1}{2i}\int_x^\infty \frac{e^{it}}{t^{\frac{3}{2}}}\,dt = \frac{e^{ix}}{i\sqrt{x}} + \frac{1}{2i}\int_x^\infty \frac{e^{it}t^{\frac{3}{2}}}{}dt$$

Iterate integration by parts to get

$$\begin{aligned} J &= +\frac{ie^{ix}}{\sqrt{x}} - \frac{1}{2}\left\{[e^{it}t^{-\frac{3}{2}}]_x^\infty + \frac{3}{2}\int_x^\infty e^{it}t^{-\frac{5}{2}}\,dt\right\} \\ &= ie^{ix}x^{-\frac{1}{2}} + \frac{1}{2}e^{ix}x^{-\frac{3}{2}} - \frac{3}{4}\left\{[-ie^{it}t^{-\frac{5}{2}}]_x^\infty - \frac{5}{2}\int_x^\infty ie^{it}t^{-\frac{7}{2}}\,dt\right\} \\ &= ie^{ix}x^{-\frac{1}{2}} + \frac{1}{2}e^{ix}x^{-\frac{3}{2}} - \frac{3}{4}ie^{ix}x^{-\frac{5}{2}} \\ &\quad +\frac{15}{8}\left\{\left[e^{it}t^{-\frac{7}{2}}\right]_x^\infty + \frac{7}{2}\int_x^\infty e^{it}t^{-\frac{9}{2}}\,dt\right\} \\ &= ie^{ix}x^{-\frac{1}{2}} + \frac{1}{2}e^{ix}x^{-\frac{3}{2}} - \frac{3}{4}ie^{ix}x^{-\frac{5}{2}} - \frac{15}{8}e^{ix}x^{-\frac{7}{2}} \end{aligned}$$

$$
\begin{aligned}
&+\frac{105}{16}\left\{[-ie^{-it}t^{-\frac{9}{2}}]_x^\infty - \frac{9}{2}\int_x^\infty ie^{it}t^{-\frac{11}{2}}\,dt\right\} \\
= \;& ie^{ix}x^{-\frac{1}{2}} + \frac{1}{2}e^{ix}x^{-\frac{3}{2}} - \frac{3}{4}ie^{ix}x^{-\frac{5}{2}} - \frac{15}{8}e^{ix}e^{-\frac{7}{2}} + \frac{105}{16}ie^{ix}x^{-\frac{9}{2}} \\
&-\frac{945}{32}\left\{[e^{it}t^{-\frac{11}{2}}]_x^\infty + \frac{11}{2}\int_x^\infty e^{it}t^{-\frac{13}{2}}\,dt\right\} \\
&\text{etc...}
\end{aligned}
$$

Now take Re part for the cos integral, Im for the sine and get

$$
\begin{aligned}
\int_x^\infty \frac{\cos t}{\sqrt{t}}\,dt \quad \sim \quad & \frac{1}{\sqrt{x}}\left\{\frac{1}{2x} - \frac{1\cdot 3\cdot 5}{(2x)^3} + \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{(2x^5)} + \cdots\right\}\cos x \\
& -\frac{1}{\sqrt{x}}\left\{1 - \frac{1\cdot 3}{(2x)^2} + \left(\frac{1\cdot 3\cdot 5\cdot 7}{2x^4}\right) + \cdots\right\}\sin x
\end{aligned}
$$

and $\int_x^\infty \frac{\sin t}{\sqrt{t}}\,dt \sim \frac{1}{\sqrt{x}}\{\cdots\}\cos x + \frac{1}{\sqrt{x}}\{\cdots\}\sin x$

The asymptotic validity is suspect! sin x and cos x have an infinite number of zeros as $x \to \infty$. Thus any truncation has an infinite number of zeros so any remainder term will also have zeros. This makes finding an implied constant difficult since you may have to show a remainder is $O(0)$! It's best to consider f and g themselves as the asymptotic expansion, extracting the oscillatory terms before bounding.