Question

(a) $\displaystyle\int_0^1 e^{ixt^3}\,dt \sim \frac{\Gamma(\frac{1}{3})e^{\frac{i\pi}{6}}}{3x^{\frac{1}{3}}},\ x \to +\infty$

(b) $\displaystyle\int_0^\infty \frac{e^{ixt^3}}{\sqrt{t}}\,dt \sim \frac{\Gamma(\frac{1}{6})e^{\frac{i\pi}{12}}}{3x^{\frac{1}{6}}},\ x \to +\infty$

Answer

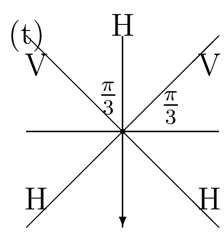
(a) $\displaystyle I + \int_0^1 e^{-xt^3}\,dt \quad x \to +\infty$

Clearly

$$\begin{aligned} h(t) &= -t^3 \\ h'(t) &= -3t^2 \Rightarrow t = 0 \text{ is a stationary point} \\ h''(t) &= -6t \\ h'''(t) &= -6 \end{aligned}$$

Locally the $h(t)$ is obviously cubic rather than quadratic. This will alter the twist angle.

Twist: $h(t=0) = 0$



Need $\begin{cases} Re(-ixt^3) < 0 \quad x \to +\infty \\ t = re^{i\theta} \\ Re(ixr^3 e^{3i\theta}) < 0 \\ \Rightarrow -\sin 3\theta < 0 \\ \sin 3\theta > 0 \end{cases}$

Now there are 3 valleys around $t = 0$ (only 2 for a quadratic case).

Angle of twist from

$$\begin{aligned} Im(-xt^3) &= 0 \\ \Rightarrow \cos 3\theta &= 0 \\ \Rightarrow 3\theta &= \frac{\pi}{2} \\ \theta &= \frac{\pi}{6} \text{ to be in the correct valley} \end{aligned}$$

Thus we have the twist:

PICTURE

$$I \sim \int_0^\infty e^{i\frac{\pi}{6}} e^{ixt^3}\, dt$$

set $\begin{array}{rcl} u & = & -it^3 \\ du & = & -3it^2\, dt \end{array}$

$$\begin{aligned} I &\sim \int_0^\infty e^{-xu} \frac{du}{-3it^2} \\ &= \int_0^\infty \frac{e^{-xu}\, du}{-3i(\frac{u}{-i})^{\frac{2}{3}}} \\ &\sim \frac{i}{3} i^{\frac{2}{3}} \int_0^\infty \frac{e^{-xu}}{u^{\frac{2}{3}}}\, du \\ &\sim \frac{(e^{i\frac{\pi}{2}})^{\frac{1}{3}}}{3} \frac{\Gamma(\frac{1}{3})}{x^{\frac{1}{3}}} \end{aligned}$$

by definition of Γ-function :

$$\int_0^\infty e^{-xt} t^{\alpha-1}\, dt = \frac{\Gamma(\alpha)}{x^\alpha}$$

$$\sim \frac{e^{\frac{i\pi}{6}}\Gamma(\frac{1}{3})}{3x^{\frac{1}{3}}} \quad x \to +\infty$$

(b) $I = \int_0^\infty \frac{e^{ixt^3}}{\sqrt{t}}\, dt$

As above, the angle of twist is $+\frac{\pi}{6}$, and we set $u = -it^3$.

$$\begin{aligned}
\Rightarrow I \quad &\sim \quad \int_0^\infty \frac{e^{-xu}}{\sqrt{t}} \frac{du}{(-3it^2)} \\
&= \quad \int_0^\infty \frac{e^{-xu}\, du}{-3i(\frac{u}{-i})^{\frac{2}{3}} \underbrace{(\frac{u}{-i})^{\frac{1}{6}}}_{=\sqrt{t}}} \\
&= \quad \frac{1}{-3i} \int_0^\infty \frac{e^{-xu}}{u^{\frac{5}{6}}}\, du(-i)^{\frac{2}{3}+\frac{1}{6}} \\
&= \quad \frac{i}{3(i)^{\frac{5}{6}}} \int_0^\infty \frac{e^{-xu}}{u^{\frac{5}{6}}}\, du \\
&= \quad \frac{(e^{i\frac{\pi}{2}})}{3} \frac{\Gamma(\frac{1}{6})}{x^{\frac{1}{6}}} \text{ by } \Gamma\text{-function definition} \\
&= \quad \frac{e^{i\frac{\pi}{2}}}{3x^{\frac{1}{6}}} \Gamma\left(\frac{1}{6}\right).
\end{aligned}$$