Question

The Bessel function of order n is defined by

$$J_n(x) = \frac{1}{\pi} ds \int_0^\pi \cos(nt - x\sin t)\, dt$$

Use the method of stationary phase to show that

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right),\ x \to +\infty.$$

Answer

$$\begin{aligned} J_n(x) &= \frac{1}{\pi}\int_0^\pi \cos(nt - x\sin t)\, dt \\ &= \frac{1}{2\pi}\int_0^\pi \left(e^{int - ix\sin t} + e^{-int + ix\sin t}\right) dt \\ &= \frac{1}{2\pi}\sum_{\pm}\int_0^\pi e^{\pm int \mp ix\sin t}\, dt \\ &= \textstyle\sum_\pm J_\pm \end{aligned}$$

Focus on $+\int -ix\sin t$ first.

We want asymptotes as $x \to +\infty,\ n$ fixed and finite

$\Rightarrow f^{(+)}(t) = e^{int},\ h^{(+)}(t) = \sin t$

$\Rightarrow h^{(+)}(t) = \cos t,\ h^{(+)''}(t) = -\sin t$

$t = \dfrac{\pi}{2}$ is the stationary point (mid-range)

$h^{(+)}\left(\dfrac{\pi}{2}\right) = +1,\ h^{(+)''}\left(\dfrac{\pi}{2}\right) = -1$

Therefore

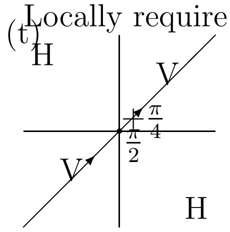
$$\begin{aligned} u^2 &= i\left[h^{(+)}(t) - h^{(+)}\left(\frac{\pi}{2}\right)\right] = i(\sin t - 1) \\ 2u\, du &= i\cos t, dt \end{aligned}$$

Locally at stationary point

$u^2 \approx i \times \dfrac{(-1)}{2!}\left(t - \dfrac{\pi}{2}\right)^2 + \cdots \approx -\dfrac{i}{2}\left(t - \dfrac{\pi}{2}\right)^2$

Twists:

Locally require



$$
\left\{
\begin{array}{l}
Re(-ix[\sin t - 1]) < 0 \quad x \to +\infty \\
Re(i[\sin t - 1]) > 0 \\
Re\left(-\frac{i}{2}\left(t - \frac{\pi}{2}\right)^2\right) > 0 \\
\Rightarrow \left(t - \frac{\pi}{2}\right) = re^{i\theta_+} \\
Re\left(-\frac{i}{2} r^2 e^{2i\theta_+}\right) > 0 \\
Re(-ie^{2i\theta_+}) > 0 \\
Re(\sin 2\theta_+) > 0
\end{array}
\right.
$$

Angle of twist from

$$
\begin{aligned}
Im\left\{i\left[h^{(+)}(t) - h^{(+)}\left(\frac{\pi}{2}\right)\right]\right\} &= 0 \\
\Rightarrow Im(-ie^{2i\theta_+}) &= 0 \\
\cos 2\theta_+ &= 0 \\
\Rightarrow \theta_+ &= \underline{\frac{\pi}{4}}
\end{aligned}
$$

PICTURE

Can use equation (3.99)

$$
\begin{aligned}
J+ &= \frac{1}{2\pi}\int_0^{\pi} e^{int-ix\sin t}\,dt \\
&\sim \frac{1}{2\pi}\sqrt{\frac{2\pi}{|xh^{(+)''}(\frac{\pi}{2})|}}e^{-ixh^{(+)}(\frac{\pi}{2})+i\theta+f^{(+)}(\frac{\pi}{2})} \\
&\sim \frac{1}{2\pi}\sqrt{\frac{\pi}{|x\times -1|}}e^{-ix+i\frac{\pi}{4}}e^{in\frac{\pi}{2}} \\
&\sim \frac{1}{2\pi}\sqrt{\frac{2\pi}{x}}e^{-i(x-\frac{n\pi}{2}-\frac{\pi}{4})}
\end{aligned}
$$

For $J_- = \frac{1}{2\pi}\int_0^{\pi} e^{-int+ix\sin t}\,dt$ the twist is in the opposite direction (due to the $+/-$ difference in $h(t) = +ix\sin t$).
Hence:

$$
h^(-)\left(\frac{\pi}{2}\right) = -1,\ h^(-)''\left(\frac{\pi}{2}\right) = +1,\ \theta^- = -\frac{\pi}{4}.
$$

Hence using (3.99)

$$
\begin{aligned}
J_- &\sim \frac{1}{2\pi}\sqrt{\frac{2\pi}{|(x\times -1)|}}e^{+ix-\frac{i\pi}{4}}e^{-in\frac{\pi}{2}} \\
&\sim \frac{1}{2\pi}\sqrt{\frac{2\pi}{x}}e^{+i(x-\frac{n\pi}{2}-\frac{\pi}{4})}
\end{aligned}
$$

To get the final answer, we add together:

$J = J_+ + J_- \sim \frac{1}{2\pi}\sqrt{\frac{2\pi}{x}}\left[e^{+i(x-\frac{n\pi}{2}-\frac{\pi}{4})} + e^{-i(x-\frac{n\pi}{2}-\frac{\pi}{4})}\right]$

$J \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$. as required.