Question

Calculate the leading order asymptotics of the following integrals as $x \to +\infty$

(a) $\displaystyle\int_0^1 \frac{e^{ix^2}}{1+t}\,dt$

(b) $\displaystyle\int_0^\pi e^{ix(\sin t+2)} t^{\frac{3}{2}} \cos(\sqrt{t+1})\,dt$

(c) $\displaystyle\int_{-1}^{+1} e^{ix\cosh t}\sqrt{1-t^2}\,dt$

Answer

(a) $\displaystyle\int_0^1 \frac{e^{ixt^2}}{(1+t)}\,dt \sim \int_0^{\infty e^{i\frac{\pi}{4}}} \frac{e^{ixt^2}}{(1+t)}\,dt$

Clearly $t = 0$ is a stationary point.

Rotate contour to get exponential decay:

$Re(ixt^2) < 0,\ Im(ixt^2) = 0$

PICTURE

Extend contour to $\infty e^{i\frac{\pi}{4}}$.

$$\begin{array}{rcl} u & = & it^2 \\ du & = & -2it\,dt \end{array} \Rightarrow t^2 = iu \Rightarrow t = e^{-\frac{\pi}{4}}\sqrt{u}$$

Therefore

$$\begin{aligned} I &\sim \int_0^\infty \frac{e^{-xu}}{(1+e^{i\frac{\pi}{4}}\sqrt{u})}\frac{du}{-2i\cdot e^{i\frac{\pi}{4}}\sqrt{u}} \\ &\sim \frac{1}{2e^{-i\frac{\pi}{4}}}\int_0^\infty \frac{e^{-xu}}{\sqrt{u}}\,du \\ &\sim \frac{e^{i\frac{\pi}{4}}}{2}\frac{\Gamma(\frac{1}{2})}{x^{\frac{1}{2}}} \\ &\sim \frac{e^{i\frac{\pi}{4}}}{2}\sqrt{\frac{\pi}{x}} \end{aligned}$$

cf. formula (3.100) $h(t) = -t^2,\ h'(t) = -2t,\ h''(t) = -2$ for all t

$\theta = +\dfrac{i\pi}{4}$ from diagram

$$\begin{aligned} I &\sim \sqrt{\frac{\pi}{2|x \times -2|}} e^{ix\cdot 0 + \frac{i\pi}{4}} \times 1 \leftarrow f(0),\ f = \tfrac{1}{1+t} \\ &\sim \frac{1}{2}\sqrt{\frac{\pi}{x}} e^{i\frac{\pi}{4}} \quad \checkmark \end{aligned}$$

(b) $J = \displaystyle\int_0^{\pi} e^{ix(\sin t + t)} t^{\frac{3}{2}} \cos(\sqrt{t+1})\, dt$

$$\left.\begin{aligned} h(t) &= -\sin t + t \\ h'(t) &= -\cos t + 1 \\ h''(t) &= \sin t \\ h'''(t) &= \cos t \end{aligned}\right\} \Rightarrow t = 0 \text{ is stationary endpoint}$$

$h(0) = 0,\ h''(0) = 0,\ h'''(0) = +1$

This ($h''(0) = 0$) is a cubic endpoint!

Too difficult! Can MA415 students see how to do it?

(c) $I = \displaystyle\int_{-1}^{+1} e^{ix\cosh t}\sqrt{1-t^2}\, dt$

$$\left.\begin{aligned} h(t) &= -\cosh t \\ h'(t) &= -\sinh t \\ h''(t) &= -\cosh t \end{aligned}\right\} \Rightarrow t = 0 \text{ is a stationary point}$$

$h(0) = h''(0) = -1$

Set $\begin{aligned} u^2 &= i[-\cosh t + 1] \\ 2u\, du &= -i \sinh t\, dt \end{aligned}$

Locally at stationary points:

$$u^2 \approx -\frac{1}{2}(t-0)^2 i = -\frac{t^2 i}{2} \Rightarrow u \approx \frac{t}{\sqrt{2}} e^{-i\frac{\pi}{4}}$$

Therefore locally rotate contour such that

$Re(ix[\cosh t - 1]) < 0$ and $Im(ix[\cosh t - 1]) = 0$

$$\Rightarrow \left.\begin{aligned} Re\left(\frac{ixt^2}{2}\right) < 0 \\ Im\left(\frac{ixt^2}{2}\right) = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x \to +\infty \\ t = |t| e^{i\frac{\pi}{4}} \end{aligned}$$

PICTURE

Extend contour to ∞.

Therefore

$$\begin{aligned}
I &\sim \int_{Re^{+i\frac{\pi}{4}}}^{Re^{i\frac{\pi}{4}}} e^{ix\cosh t}\sqrt{1-t^2}\,dt \\
&\sim e^{ix}\int_{-\infty}^{+\infty} e^{-xu^2}\frac{2u\sqrt{1-t^2}}{-i\sinh t}\,du \quad x\to+\infty \\
&\sim e^{ix}\int_{-\infty}^{+\infty} e^{-xu^2}\frac{2u\sqrt{1-2iu^2}}{-i\sqrt{2}e^{i\frac{\pi}{4}}u}\,du \quad x\to+\infty \\
&\sim e^{ix}\int_{-\infty}^{+\infty} e^{ixu^2}\,du\sqrt{2}e^{i\frac{\pi}{4}} \quad x\to+\infty \\
&\sim e^{ix+i\frac{\pi}{4}}\sqrt{\frac{\pi}{x}}\sqrt{2} \\
&\sim \sqrt{\frac{2\pi}{x}}e^{ix+i\frac{\pi}{4}} \quad x\to+\infty
\end{aligned}$$

Check with (3.99)

$$I \sim \sqrt{\frac{2\pi}{|xh''(0)|}}e^{-ixh(0)+i\theta}f(0) \quad x\to+\infty$$

$h(0)=-1,\ h''(0)=-1,\ f(0)=\sqrt{1}=1,\ \theta=+\frac{\pi}{4}$ from diagram

$$\sim \sqrt{\frac{2\pi}{x}}e^{ix+i\frac{\pi}{4}} \quad \checkmark$$