Question

The modified Bessel function of the first kind may be defined by

$$I_n(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos t}\cos(nt)\,dt.$$

Show that

$$I_n(x) \sim \frac{e^x}{\sqrt{2\pi x}},\ x \to +\infty.$$

Answer

$I_n(x) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} e^{x\cos t}\cos(nt)\,dt$

On the interval $[0, \pi]$, the max. value of cos t (min. value of $-\cos t$) is at the endpoint $t = 0$.

Therefore set

$u = h(t) - h(0) = -\cos t + 1 \;\; du = \sin t\,dt$

Thus

$I_n(x) = \dfrac{1}{\pi}\displaystyle\int_0^{2} e^{-xu+x}\frac{\cos[(nt(u))]}{\sin[t(u)]}\,du$

$u \approx h''(0)\dfrac{(t-0)^2}{2} + \cdots \Rightarrow u \approx \dfrac{t^2}{2^2}$

$$\begin{aligned}
I_n(x) &\approx \frac{e^x}{\pi}\int_0^{2} e^{-xu}\frac{\cos(n\sqrt{2u})}{\sin(\sqrt{2u})}\,du \\
&\sim \frac{e^x}{\pi}\int_0^{\infty} e^{-xu}\frac{\cos(n\sqrt{2u})}{\sin(\sqrt{2u})}\,du \quad x \to +\infty \\
&\quad \frac{\cos(n\sqrt{2u})}{\sin(\sqrt{2u})} \\
&\quad \sim \frac{1 - \frac{n^2 2u}{2} + \cdots}{\sqrt{2u} - \frac{2\sqrt{2}u^{\frac{3}{2}}}{6} + \cdots} \text{ as u} \to 0 \text{ and hence} \sim \frac{1}{\sqrt{2\text{u}}} \\
&\sim \frac{e^x}{\pi}\int_0^{\infty}\frac{e^{-xu}}{\sqrt{2}\sqrt{u}}\,du \\
&\sim \frac{e^x}{\pi}\frac{\Gamma(+\frac{1}{2})}{\sqrt{2}\sqrt{x}} \\
&\sim \frac{e^x}{\sqrt{2\pi x}} \quad x \to +\infty
\end{aligned}$$