Question

This question requires some ingenuity, rather than any standard method.

Remembering that

$$\left(1+\frac{1}{t}\right)^t = \exp\left(t\log\left[1+\frac{1}{t}\right]\right)$$

and expanding the integral as a power series in $\frac{1}{t}$ which is convergent for $t>1$, show that

$$\int_1^x \left(1+\frac{1}{t}\right)^t dt \sim xe - \frac{1}{2}e\log x + O(1)$$

Answer

$\left(1+\frac{1}{t}\right)^t = e^{t\log(1+\frac{1}{t})}$

$\int_1^x \left(1+\frac{1}{t}\right)^t dt = \int_1^x e^{t\log(1+\frac{1}{t})}\,dt$

Expand

$$\begin{aligned} t\log\left(1+\frac{1}{t}\right) &= t\left[\frac{1}{t}-\frac{1}{2t^2}+\frac{1}{3t^3}-\cdots\right] \\ & \underline{\text{convergent}}\ \text{for}|t|>1(|frac1t|<1) \\ &= 1-\frac{1}{2t}+\frac{1}{3t^2}-\cdots \end{aligned}$$

Therefore

$$\begin{aligned} e^{t\log(1+\frac{1}{t})} &= e^{1-\frac{1}{2t}+\frac{1}{3t^2}-\cdots} \\ &= e\cdot e^{-\frac{1}{2t}+\frac{1}{3t^2}}-\cdots \\ &= e\left[1-\frac{1}{2t}+\frac{1}{3t^2}-\cdots+\frac{1}{2!}\left(-\frac{1}{2t}+\frac{1}{3t^2}+\cdots\right)^2+\cdots\right] \\ &= e\left[1-\frac{1}{2t}+O\left(\frac{1}{t^2}\right)\right] \text{ for |t|} > 1 \end{aligned}$$

Thus

$$\begin{aligned} \int_1^x \left(1+\frac{1}{t}\right)^t dt &= e\int_1^x \left[1-\frac{1}{2t}+O\left(\frac{1}{t^2}\right)\right] dt \\ &= e\left[t-\frac{1}{2}\log t+O\left(\frac{1}{t}\right)\right]_1^x \\ &= e\left[x-\frac{1}{2}\log x+O91)\right] \text{ as x} \to +\infty \\ &\sim ex-\frac{1}{2}e\log x \text{ as x} \to +\infty \quad \text{as required.} \end{aligned}$$