Question
Consider the integral

$$\int_0^\infty e^{-xt}e^{-\frac{1}{t}}\,dt,\ x \to +\infty$$

At first sight, it might seem that a direct application of Watson's lemma should produce an asymptotic expansion for the integral. Why is this not so? Use the change of variables $t\sqrt{x} = s$ to show that

$$\int_0^\infty e^{-xt}e^{-\frac{1}{t}}\,dt \sim x^{-\frac{3}{4}}\sqrt{\pi}e^{-2\sqrt{x}} \quad x \to +\infty$$

Answer
$\int_0^z inftye^{-xt}e^{-\frac{1}{t}}\,dt \quad x \to +\infty$
Watson's lemma requires $e^{-\frac{1}{t}}$ to be represented by an asymptotic power series, with $e^{-\frac{1}{t}} \sim a_0 t^{\lambda_0}$ (where $\lambda_0 > -1$) as $t \to 0^+$, say, as the dominant term.
But $\lim_{t\to 0^+} t^{-\lambda_0}e^{-\frac{1}{t}} = \lim_{\tau\to+\infty} \tau^{\lambda_0}e^{-\tau} = 0(\tau = \frac{1}{t})$ for all λ_0
Consequently no such expansion exists and we cannot use Watson's lemma: $e^{-\frac{1}{t}}$ tends to zero faster than any power of t as $t \to 0^+$ and therefore has no power series expansion about $t = 0$.
Use change of variable given:$t\sqrt{x} = s$
$\Rightarrow ds = \sqrt{x}\,dt$, so for $x > 0$, the integral becomes

$$\begin{aligned} I &= \int e^{-xt}e^{-\frac{1}{t}}\,dt \\ &= \frac{1}{\sqrt{x}}\int_0^\infty e^{x\frac{s}{\sqrt{x}}}e^{-\frac{\sqrt{x}}{s}}\,ds \\ &= \frac{1}{\sqrt{x}}\int_0^\infty e^{-\sqrt{x}(s+\frac{1}{s})}\,ds \end{aligned}$$

We're now in the Laplace ball-park with large parameter $\sqrt{x}$

$$h(s) = s + \frac{1}{s} \Rightarrow h'(s) = 1 - \frac{1}{s^2} \Rightarrow s = \pm 1 \text{ are min/max}$$

For $0 \le s < \infty,\ s = +1$ is the critical point.
$h''(t) = +2$ so it's a min.

Thus we use

$$\begin{aligned} u^2 &= h(s) - h(1) = h(s) + 2 \\ 2u\,du &= h'(s)\,ds \\ u^2 &\approx \frac{h''(1)}{2}(s-1)^2 = (s-1)^2 \end{aligned}$$

so $u \approx s - 1,\ h'(s) = h''(1)(s-1) = 2(s-1) \approx 2u$

Therefore

$$\begin{aligned} I &= \frac{1}{\sqrt{x}} \int_{-\sqrt{h(0)-h(1)}}^{+\infty} e^{-\sqrt{x}u^2 - 2\sqrt{x}} \frac{2u\,du}{h'(s(u))} \\ &\sim \frac{2e^{-2\sqrt{x}}}{\sqrt{x}} \int_{-\infty}^{+\infty} \frac{u\,du}{2u} \\ &\sim \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \int_{-\infty}^{+\infty} e^{-\sqrt{x}u^2}\,du \\ &\sim \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \sqrt{\frac{\pi}{\sqrt{x}}} \\ &\sim \frac{e^{-2\sqrt{x}}\sqrt{\pi}}{x^{\frac{3}{4}}} \end{aligned}$$