Question
Use Stirling's approximation to show that

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim x^{a-b}, \ x \to +\infty$$

Answer
Stirling's formula is $\Gamma(x) \sim \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x} \quad x \to +\infty$
so substitute directly:

$$\begin{aligned}
\frac{\Gamma(x+a)}{\Gamma(x+b)} &\sim \frac{\sqrt{2\pi}(x+a)^{(x+a-\frac{1}{2})}e^{-x-a}}{\sqrt{2\pi}(x+b)^{(x+b-\frac{1}{2})}e^{-x-b}} \ x \to +\infty \\
&\sim \left(\frac{x+a}{a+b}\right)^x e^{b-a} \frac{(x+a)^{a-\frac{1}{2}}}{(x+b)^{b-\frac{1}{2}}} \\
&\sim \frac{(1+\frac{a}{x})^x}{(1+\frac{b}{x})^x} e^{b-a} x^{a-b} \frac{(1+\frac{a}{x})^{a-\frac{1}{2}}}{(1+\frac{b}{x})^{b-\frac{1}{2}}}
\end{aligned}$$

Now $\left(1+\frac{a}{x}\right)^x \to e^a$ as $x \to +\infty$ (by definition of e^a).
Likewise $\left(1+\frac{b}{x}\right)^x \to e^b$

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim \frac{e^a}{e^b} \cdot \frac{e^b}{e^a} x^{a-b} \underbrace{\frac{(1+\frac{a}{x})^{a-\frac{1}{2}}}{(1+\frac{b}{x})^{b-\frac{1}{2}}}}$$

Therefore

$$= \frac{1^{a-\frac{1}{2}}}{1^{b-\frac{1}{2}}} \text{ as x} \to +\infty$$

$$\sim \quad x^{a-b} \quad \text{as x} \to +\infty$$

Hence $\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim x^{a-b}$ as x $\to$ + infty as required.