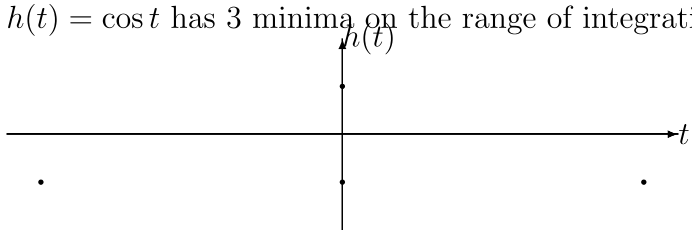
Question
Show that

$$\int_{-2\pi}^{2\pi} (1+t)e^{x\cos t}\,dt \sim 2e^x\sqrt{\frac{2\pi}{x}},\ x \to +\infty$$

and obtain one further term in the asymptotic expansion.

Answer
$I = \displaystyle\int_{-2\pi}^{+2\pi} (1+t)e^{x\cos t}\,dt$
$h(t) = \cos t$ has 3 minima on the range of integration at $t = 0,\ \pm 2\pi$



Now we see that $h(0) = h(\pm 2\pi)$ so it looks like all 3 will give equally dominant contributions. In fact we have 2 quadratic endpoints and one quadratic minimum. If we now consider the periodicity of $h9t)$, we see

$$h(t \pm 2\pi) = -\cos(2 \pm 2\pi) = -\cos t = h(t)$$

Thus the two endpoints add up to give one <u>full</u> contribution identical to the one at $t = 0$. Thus we focus on $t = 0$ and just double its contribution to get the full result.

Set $\begin{aligned} u^2 &= h(t) - h(0) \\ h(t) &= -1 + \frac{(t=0)^2}{2} + \cdots \end{aligned}$

$$\begin{aligned} h(t) &= -\cos t \\ h'(t) &= +\sin t \\ h''(t) &= +\cos t \end{aligned}$$

Therefore $u^2 \approx \dfrac{t^2}{2},\ u \approx \dfrac{t}{\sqrt{2}}$
and $2u\,du = h'(t)\,dt,\ h'(t) \approx h''(0)\,t$
Therefore

$$\begin{aligned}
I &= 2\int_{-\sqrt{h(-2\pi)-h(0)}}^{\sqrt{h(2\pi)-h(0)}} \frac{u(1+t(u))}{h'(t(u))} e^{-xu^2-xh(0)}\, du \\
&\sim 2e^x \int_{-\infty}^{+\infty} e^{-xu^2} \frac{u(1+t(u))}{h'(t(u))}\, du \\
&\sim 2e^x \int_{-infty}^{+\infty} e^{-xu^2} \frac{u}{h''(0)\sqrt{2}u}\, du \\
&\sim \sqrt{2}e^x \int_{-\infty}^{+\infty} e^{-xu^2}\, du \\
&\sim \sqrt{\frac{2\pi}{x}} e^x \quad x \to +\infty
\end{aligned}$$

Thus doubling up the contribution (including endpoints) we have (as required)

$$I \sim 2e^x \sqrt{\frac{2\pi}{x}} \quad x \to +\infty$$

To get one further term in the expansion we need to retain more terms in the expansion of

$$\begin{aligned}
\frac{u(1+t)}{h'(t)} &= \frac{u(1+t)}{\sin t} \quad \text{about t} = 0 \\
&= \frac{u(1+t)}{(t - \frac{t^3}{3!} + \cdots)} \\
&\approx \frac{u}{t}(1+t)\left(1 - \frac{t^2}{6} + \cdots\right)^{-1} \\
&\approx \frac{u}{t}(1+t)\left(1 + \frac{t^2}{6} + \cdots\right) \\
&= \frac{u}{t}\left(1 + t + \frac{t^2}{6} + O(t^3)\right) \quad (A)
\end{aligned}$$

Now also need u to higher order in t:
$u^2 = h(t) = h(0) = \dfrac{h''(0)}{2!}(t-0)^2 + \dfrac{h'''(0)}{3!}(t-0)^3 = \dfrac{h^{iv}}{4!}(t-0)^4 + \cdots$
so from above $u^2 = \dfrac{t^2}{2} - \dfrac{t^4}{24} + \cdots$
so $u = \dfrac{t}{\sqrt{2}}\left(1 - \dfrac{t^2}{12} + \cdots\right)^{\frac{1}{2}} \approx \dfrac{t}{\sqrt{2}}\left(1 - \dfrac{t^2}{24}\right) \quad (B)$
Therefore putting (B) in (A):

$$\frac{u(1+t)}{h'(t)} \approx \frac{t}{\sqrt{2}} \frac{(1 - \frac{t^2}{24})}{t} \left(1 + t + \frac{t^2}{6} + \cdots\right)$$

$$
\begin{aligned}
&\approx \frac{1}{\sqrt{2}}\left(1-\frac{t^2}{24}\right)\left(1+t+\frac{t^2}{6}\right) \\
&= \frac{1}{\sqrt{2}}\left(1+t+\frac{t^2}{8}\right)
\end{aligned}
$$

Thus we have

$$\frac{u(1+t)}{h'(t)} \approx \frac{1}{\sqrt{2}}\left(1+\sqrt{u}+\frac{2u^2}{8}\right)$$

Substitution into original integral gives:

$$I \sim 2e^x \int_{-\infty}^{+\infty} \frac{e-xu^2}{\sqrt{2}}\left(1+\sqrt{2}+\frac{2u^2}{8}\right)$$

First term gives $\sqrt{\frac{2\pi}{x}}e^x$ as before.

Second term is ZERO: $\int_{-\infty}^{+\infty} e^{-xu^2}\ \underset{\substack{\uparrow \\ \text{odd function}}}{u}\, du = 0$

Third term is $\frac{\sqrt{2}e^x}{4} \underbrace{\int_{-\infty}^{+\infty} e^{-xu^2}\ u^2\, du}_{= -\frac{\partial}{\partial x}\int_{-\infty}^{+\infty} e^{-xu^2}\, du} = \frac{e^x}{2\sqrt{2}}\frac{1}{2x}\sqrt{\frac{\pi}{x}}$

Thus pulling everything together and multiplying by 2 for the 2 endpoint contributions, we have

$$I \sim 2e^x\sqrt{\frac{2\pi}{x}}\left(1+\frac{1}{8x}+O\left(\frac{1}{x^2}\right)\right)$$

<u>Phew!!!</u>