QUESTION

Let S behave lognormally such that $ds = S(\mu dt + \sigma dW)$. Write down Itô's lemma for a general (deterministic) $f(S)$. Hence find the stochastic differential equations satisfied by:

(i) $f(S) = As + B$;

(ii) $g(S) = S^n$,

where $A,\ B,\ n$ are constants.

ANSWER

$$ds = S(\mu dt + \sigma dw)$$

$$\begin{aligned}
df &= \frac{\partial f}{\partial S}ds + \frac{1}{2}\frac{\partial^2 f}{ds^2}(ds)^2 \text{ no time dependence} \\
df &= \frac{\partial f}{\partial S}(S\mu dt + S\sigma dw) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(S^2(\mu dt + \sigma dw)^2) \\
&= f'S\mu dt + f'S\sigma dw + \frac{1}{2}f''S^2(\mu(dt)^2 + 2\mu dtdw\sigma + \sigma^2(dw)^2)
\end{aligned}$$

But rule of thumb $\Rightarrow (dt)^2 = 0,\ (dtdw) =)$ (i.e. negligible with respect to dt,) and$(dw)^2 = dt$. Therefore

$$\begin{aligned}
df &= f'S\mu dt + f'S\sigma dw + \frac{1}{„}f''S^2\sigma^2 dt \\
\text{or } df &= f'S\sigma dw + (f'S\mu + \frac{1}{2}f''S^2\sigma^2)dt
\end{aligned}$$

(i)

$$\begin{aligned}
f(S) &= As + B,\ f' = A,\ f'' = 0,\ As = f - B \\
df &= As\sigma dw + As\mu dt \\
df &= (f - B)[\sigma dw + \mu dt]
\end{aligned}$$

(ii)

$$\begin{aligned}
g(S) &= S^n \Rightarrow g' = ns^{n-1},\ g'' = n(n-1)S^{n-2} \\
dg &= ns^{n-1}.S\sigma dw + \left[ns^{n-1}S\mu + \frac{n(n-1)S^{n-2}S^2\sigma^2}{2}\right]dt \\
dg &= g\left[n\sigma dw + \left(n\mu + \frac{n(n-1)}{2}\sigma^2\right)dt\right] \text{ as } g = S^n \\
\text{or } dg &= ng\left[\sigma dw + \left(\mu + \frac{(n-1)}{2}\sigma^2\right)dt\right]
\end{aligned}$$