Question

An aircraft has speed v and a flying range (out and back) of R in calm weather. Prove that in a north wind (i.e. from the north) of speed n $(n < v)$ its range is

$$\frac{R(v^2 - n^2)}{v(v^2 - n^2\sin^2\phi)^{\frac{1}{2}}}$$

in a direction whose true bearing from north is ϕ.

What is the maximum value of this range and it what directions may it be attained?

Answer

The range of the plane is determined by its flying time. In still air flying time is $2RV^{-1}$. Hence when there is a wind the total time is $2RV^{-1}$.

I.e. $T_{\text{out}} + T_{\text{in}} = 2RV^{-1}$

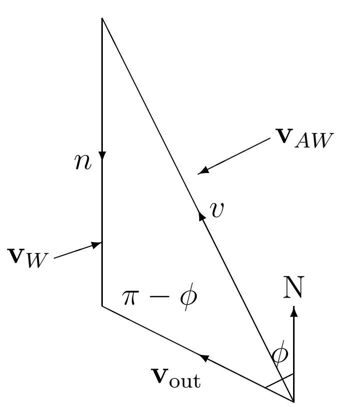
where T_{out} and T_{in} are the times for the outward and inward legs.

Now if the range in the wind is R_ϕ then $T_{\text{out}} = \frac{R}{v_{\text{out}}}$ and $T_{\text{in}} = \frac{R}{v_{\text{in}}}$ where v_{out} and v_{in} are outward and inward speeds relative to the ground.

<u>Outward trip</u>

$\mathbf{v}_{\text{out}} = \mathbf{v}_{AW} + \mathbf{v}_W$

where $\mathbf{v}_{AW}$ is the velocity of the aeroplane relative to the wind and $\mathbf{v}_W$ is the velocity of the wind relative to the ground.



$$\begin{aligned}
\text{cosine rule } v^2 &= v_{\text{out}}^2 + n^2 - 2nv_{\text{out}}\cos(\pi - \phi) \\
\Rightarrow 0 &= v_{\text{out}}^2 + 2n\cos\phi v_{\text{out}} - v^2 \\
\Rightarrow v_{\text{out}} &= \frac{-2n\cos\phi \pm \sqrt{4n^2\cos^2\phi - 4n^2 + 4v^2}}{2}
\end{aligned}$$

$$\Rightarrow v_{\text{out}} \quad = \quad -n\cos\phi + \sqrt{n^2\cos^2\phi - n^2 + v^2}$$

A similar calculation for outward trip gives

$$v_{\text{in}} = n\cos\phi + \sqrt{v^2 - n^2\sin^2\phi}$$

$$\Rightarrow \frac{R_\phi}{v_{\text{in}}} + \frac{R_\phi}{v_{\text{out}}}$$

$$= \left[\frac{1}{-n\cos\phi + \sqrt{v^2 - n^2\sin^2\phi}} + \frac{1}{n\cos\phi + \sqrt{v^2 - n^2\sin^2\phi}}\right] R_\phi = \frac{2R}{v}$$

$$\Rightarrow R_\phi = \frac{R(v^2 - n^2)}{v\sqrt{v^2 - n^2\sin^2\phi}}$$

By inspection R_ϕ maximum when the denominator in minimum at $\phi = \pm\frac{\pi}{2}$ in which case the maximum range is $R\left(1 - \frac{n^2}{v^2}\right)$ in headings due east or west.