

Question

Find the values of λ and μ for which the following systems of equations is consistent, and find the general solution in all cases.

$$\begin{aligned}x_1 + 2x_2 + x_3 + x_4 &= 13 \\x_1 + \lambda x_2 + 4x_3 + 6x_4 &= 18 \\x_1 - 5x_3 - 9x_4 &= \lambda \\x_1 + 5x_2 + 10x_3 + 16x_4 &= \mu\end{aligned}$$

Answer

$$\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 13 \\ 1 & \lambda & 4 & 6 & 18 \\ 1 & 0 & -5 & -9 & \lambda \\ 1 & 5 & 10 & 16 & \mu \end{array} \rightarrow \begin{array}{cccc|c} 1 & 2 & 1 & 1 & 13 \\ 0 & \lambda - 2 & 3 & 5 & 5 \\ 0 & -2 & -6 & -10 & \lambda - 13 \\ 0 & 3 & 9 & 15 & \mu - 13 \end{array}$$

$$\rightarrow \begin{array}{cccc|c} 1 & 2 & 1 & 1 & 13 \\ 0 & \lambda - 2 & 3 & 5 & 5 \\ 0 & 3 & 9 & 15 & -\frac{3}{2}(\lambda - 13) \\ 0 & 3 & 9 & 15 & \mu - 13 \end{array}$$

Must have $\mu - 13 = -\frac{3}{2}(\lambda - 13)$

i.e. $3\lambda + 2\mu - 65 = 0$

Then we get rid of row 4

$$\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 13 \\ 0 & \lambda - 2 & 3 & 5 & 5 \\ 0 & 3 & 9 & 15 & -\frac{3}{2}(\lambda - 13) \end{array}$$

$\lambda = 3$ gives both sides of row 3 to be zero.

$\lambda \neq 3$ gives $x_2 = \frac{1}{2}$

$\lambda = 3$ gives $\mu = 28$

Thus
$$\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 13 \\ & 1 & 3 & 5 & 5 \end{array}$$

Solution is

$$\begin{pmatrix} 3 \\ 5 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 5 \\ -3 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 9 \\ -5 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda \neq 3$ gives $x_2 = \frac{1}{2}$ and solution

$$\begin{pmatrix} 10 + \frac{\lambda}{6} \\ \frac{1}{2} \\ 2 - \frac{\lambda}{6} \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \frac{2}{3} \\ 0 \\ -\frac{5}{3} \\ 1 \end{pmatrix}$$