Question

(a) Show that all the roots of the equation

$$(1+z)^{2n+1} = (1-z)^{2n+1}$$

are given by

$$\pm i \tan\left(\frac{k\pi}{2n+1}\right) \qquad k = 0, 1, 2, \cdots, n$$

(b) Let $z = x + iy$ and $w = u + iv$. If $w = z^2 + 2z$ show that the line $v = 2$ is the image of a rectangular hyperbola in the z-plane. Sketch this hyperbola.

(c) if $w = az + b$, where $a = 3(1-i)$ and $b = 2 + 3i$, then describe what happens to any figure in the z-plane under this transformation.

Answer

(a)

$$\begin{aligned}
(1+x)^{2n+1} &= (1-x)^{2n+1} \\
\text{So} \quad \frac{1+x}{1-x} &= = e^{\frac{2\pi i}{2n+1}k} \\
x &= \frac{e^{\frac{2\pi i}{2n+1}k} - 1}{e^{\frac{2\pi i}{2n+1}k} + 1} \\
&= \frac{e^{\frac{\pi i k}{2n+1}} - e^{-\frac{\pi i k}{2n+1}}}{e^{\frac{\pi i k}{2n+1}} + e^{-\frac{\pi i k}{2n+1}}} \\
&= i \tan\frac{\pi k}{2n+1} \qquad k = -n, \cdots, n \\
&= \pm i \tan\frac{\pi k}{2n+1} \qquad k = 0, \cdots, n
\end{aligned}$$

(b) $z = x + iy$

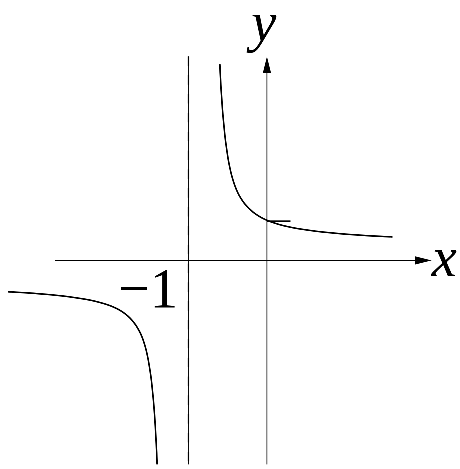
$w = u + iv$

Therefore as $w = z^2 + 2z$, $u + iv = x^2 - y^2 + 2ixy + 2(x + iy)$

S $v = 2xy + 2x$

Thus $v = 2$ if and only if $2xy + 2x = 2$ and $y(x+1) = 1$

This is a rectangular hyperbola.



(c)

$$\begin{aligned} w &= 3(1-i)z + (2+3i) \\ &= 3\sqrt{2}e^{-\frac{i\pi}{4}}z + (2+3i) \end{aligned}$$

So any figure is rotated clockwise through 45°, magnified by $3\sqrt{3}$ and the translated by $(2+3i)$