

Question

(a) Reduce the conic

$$6x^2 - 24xy - y^2 + 60x - 20y + 45 = 0$$

to standard form, and identify its type.

(b) Find the eccentricity, foci, and semi latus rectum of the ellipse

$$3x^2 + 4y^2 = 1$$

Hence write down all its polar equation referred to a focus as origin and major axis as initial line.

Answer

(a) $6x^2 - 24xy - y^2 + 60x - 20y + 45 = 0$

Let $x = \xi + h$ and $y = \eta + k$

$$6(\xi + h)^2 - 24(\xi + h)(\eta + k) - (\eta + k)^2 + 60(\xi + h) - 20(\eta + k) + 45 = 0$$

$$6\xi^2 - 24\xi\eta - \eta^2 + \xi(12h - 24k + 60) - \eta(24h + 2k + 20) + 6h^2 - 24hk - k^2 + 60h - 29k + 45 = 0$$

Choose h, k so that

$$\left. \begin{array}{l} 12h - 24k + 60 \\ 24h + 2k + 20 \end{array} \right\} k = 2, h = -1$$

So the equation becomes

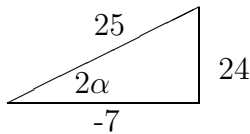
$$6\xi^2 - 24\xi\eta - \eta^2 = 5$$

Now $\xi = X \cos \alpha - Y \sin \alpha$ and $\eta = X \sin \alpha + Y \cos \alpha$ gives

$$\begin{aligned} 5 &= 6(X^2 \cos^2 \alpha - 2XY \cos \alpha \sin \alpha + Y^2 \sin^2 \alpha \\ &\quad - 24(X \cos \alpha - Y \sin \alpha)(X \sin \alpha + Y \cos \alpha) \\ &\quad - (X^2 \sin^2 \alpha + 2XY \cos \alpha \sin \alpha + Y^2 \cos^2 \alpha) \end{aligned}$$

$$\begin{aligned} 5 &= X^2(6 \cos^2 \alpha - 24 \cos \alpha \sin \alpha - \sin^2 \alpha) \\ &\quad Y^2(6 \sin^2 \alpha + 24 \cos \alpha \sin \alpha - \cos^2 \alpha) \\ &\quad XY(-12 \cos \alpha \sin \alpha - 24(\cos^2 \alpha - \sin^2 \alpha) - 2(\cos \alpha \sin \alpha)) \end{aligned}$$

We want the coefficient of XY to be zero $-24 \cos 2\alpha - 7 \sin 2\alpha = 0$ So
 $\tan 2\alpha = -\frac{24}{7}$



Choose $\cos 2\alpha = -\frac{7}{25}$ and $\sin \alpha = \frac{24}{25}$
 $\cos^2 \alpha = \frac{1}{2}(\cos 2\alpha + 1) = \frac{9}{25}$ $\sin^2 \alpha = \frac{16}{25}$

$$3Y^2 - 2X^2 = 1$$

(b) Eccentricity = $\frac{1}{2}$

$$\text{Foci} = \left(\pm \frac{1}{2\sqrt{3}}, 0 \right)$$

$$l = \frac{\sqrt{3}}{4}$$

$$\text{Polar equation } \frac{\sqrt{3}}{4r} = 1 \pm \frac{1}{2} \cos \theta$$