Question
Consider the following set of simultaneous equations

$$\begin{aligned} kx + y - 2z &= 3 \\ x - y + 2z &= 1 \\ -x + 2y + z &= 2 \end{aligned}$$

If $k = 0$, find the solution by matrix inversion.
Note: if you fail to show detailed working of the matrix inversion, no marks will be awarded, even if you can write down the correct answer.

Answer
$k = 0$

$$\begin{aligned} y - 2z &= 3 \\ x - y + 2z &= 1 \\ -x + 2y + z &= 2 \end{aligned}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 & -2 \\ 1 & -1 & 2 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$\mathbf{A} \cdot \quad \mathbf{X} \quad = \quad \mathbf{K}$

Require $\mathbf{A}^{-1}$, since $\mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$

Step (iii)

$$\begin{aligned} \triangle &= detA \\ &= \begin{vmatrix} 0 & 1 & -2 \\ +1 & -1 & 2 \\ -1 & 2 & 1 \end{vmatrix} \\ &= 0\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} \\ &= -1 \times (1+2) - 2 \times (2-1) \\ &= -3 - 2 \\ &= \underline{-5} \end{aligned}$$

so solution exists

Step (i) Cofactors of matrix are given by

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 \\ 1 & -1 & 2 \\ -1 & 2 & 1 \end{pmatrix}$$

cofactor of $A_{11} = + \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = -5$ Following + - sign pattern

cofactor of $A_{12} = - \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = -3$

cofactor of $A_{13} = + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = +1$

cofactor of $A_{21} = - \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = -5$

cofactor of $A_{22} = + \begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix} = -2$

cofactor of $A_{23} = - \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = -1$

cofactor of $A_{31} = + \begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} = 0$

cofactor of $A_{32} = - \begin{vmatrix} 0 & -2 \\ 1 & 2 \end{vmatrix} = -2$

cofactor of $A_{33} = + \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$

Matrix of cofactors is thus:

$$\begin{pmatrix} -5 & -3 & 1 \\ -5 & -2 & -1 \\ 0 & -2 & -1 \end{pmatrix}$$

Step (ii)

Transpose this to get $adj A$

$$adj\ A = \begin{pmatrix} -5 & -3 & 1 \\ -5 & -2 & -1 \\ 0 & -2 & -1 \end{pmatrix}^T = \begin{pmatrix} -5 & -5 & 0 \\ -3 & -2 & -2 \\ 1 & -1 & -1 \end{pmatrix}$$

Step (iv)

$A^{-1} = \dfrac{adj\ A}{det\ A} = \dfrac{1}{-5} \begin{pmatrix} -5 & -5 & 0 \\ -3 & -2 & -2 \\ 1 & -1 & -1 \end{pmatrix}$

Hence $\mathbf{A} = -\dfrac{1}{2} \begin{pmatrix} -5 & -5 & 0 \\ -3 & -2 & -2 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

$\Rightarrow \underline{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\dfrac{1}{5} \begin{pmatrix} -20 \\ -15 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}}$