Question

(i) Convert the cartesian equation of the plane

$$x - 2y + 2z = 5$$

into a vector equation of the form $\mathbf{r} \cdot \hat{\mathbf{n}} = \mathbf{d}$, where $\hat{\mathbf{n}}$ is a unit vector.

What is the geometric significance of $\hat{\mathbf{n}}$?

State the perpendicular distance of the plane from the origin.

(ii) Find the intersections (if any) of the above plane with the following lines

(a) $\mathbf{r} = -5\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$

(b) $\mathbf{r} = 6\mathbf{i} + \mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} + \mathbf{j})$

Comment on result of (b).

Answer

(i) $x - 2y + 2z = 5$

Set $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and we have

$$\mathbf{r} \cdot (\underbrace{\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}}) = 5$$

This is not a unit vector

so

$$\begin{aligned} \hat{\mathbf{n}} &= \frac{\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}}{|\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}|} \\ &= \frac{\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}}{\sqrt{1+4+4}} \\ &= \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \end{aligned}$$

so divide both sides by 3 and get

$$\mathbf{r} \cdot \underbrace{\left(\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right)} = \underbrace{\frac{5}{3}}$$

$\hat{\mathbf{r}}$ d

$\hat{\mathbf{n}}$ is <u>a unit normal to the plane</u>

d is the perpendicular distance of the plane from the origin.

So here $d = \dfrac{5}{3}$

(ii) **(a)** We have

$$[-\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})] \cdot \left[\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right] = \frac{5}{3}$$

$$\Rightarrow (-5 + \lambda, 3 - 2\lambda, -1 + 2\lambda) \cdot \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right) = \frac{5}{3}$$

$$\Rightarrow \frac{-5 + \lambda}{3} - \frac{2}{3}(3 - 2\lambda) + \frac{2}{3}(-1 + 2\lambda) = \frac{5}{3}$$

$$\Rightarrow -5 + \lambda - 6 + 4\lambda - 2 + 4\lambda = 5$$

$$\Rightarrow 9\lambda = 18$$

$$\Rightarrow \underline{\lambda = 2}$$

So there exists a single intersection point of line and plane at

$$\begin{aligned} \mathbf{r} &= -5\mathbf{i} + 3\mathbf{j} - \mathbf{k} + 2(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \\ &= \underline{-3\mathbf{i} - \mathbf{j} + 3\mathbf{k}} \end{aligned}$$

(b) We have

$$[(6 + 2\mu)\mathbf{i} + (1 + \mu)\mathbf{j} + \mathbf{k}] \cdot \left[\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right] = \frac{5}{3}$$

Therefore $6 + 2\mu - 2(1 + \mu) + 2 = 5$

Therefore $6 + 2\mu - 2 - 2\mu + 2 = 5$

$\Rightarrow 6 = 5$!!

Therefore no value of μ can satisfy this equation so no intersection.
Line must be parallel to plane.