Question

Let the vertices of a triangle ABC have the following position vectors relative to some origin O:

$$\begin{aligned} \mathbf{OA} &= \mathbf{i} + 2\mathbf{j} + \mathbf{k}, \\ \mathbf{OB} &= 2\mathbf{i} + \mathbf{j}, \\ \mathbf{OC} &= \mathbf{i} - \mathbf{j} + 2\mathbf{k}. \end{aligned}$$

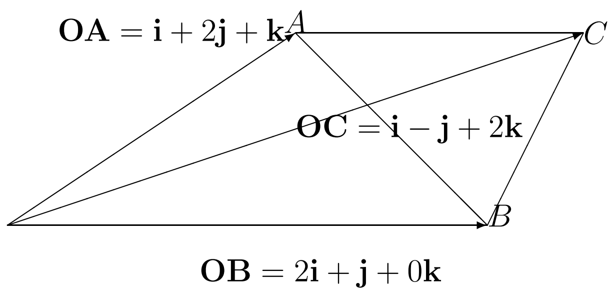
(i) Show these vectors and the triangle on a rough sketch.

(ii) Find the angle between **AB** and **AC**. Repeat the calculation for **BA** and **BC**. Hence deduce the three angles within the triangle.

(iii) Calculate the area of the triangle ABC using an appropriate vector product.

Answer

(i)



(or topologically equivalent...)

(ii)

$$\begin{aligned} \mathbf{AB} = \mathbf{AO} + \mathbf{OB} &= -\mathbf{OA} + \mathbf{OB} \\ &= -\mathbf{i} - 2\mathbf{j} - \mathbf{k} + 2\mathbf{i} + \mathbf{j} \\ &= \mathbf{i} - \mathbf{j} - \mathbf{k} \end{aligned}$$

$$\begin{aligned} \mathbf{AC} = \mathbf{OC} - \mathbf{OA} &= \mathbf{i} - \mathbf{j} + 2\mathbf{k} - \mathbf{i} - 2\mathbf{j} - \mathbf{k} \\ &= -3\mathbf{j} + \mathbf{k} \end{aligned}$$

$$\begin{aligned}\mathbf{BC} = \mathbf{OC} - \mathbf{OB} &= \mathbf{i} - \mathbf{j} + 2\mathbf{k} - 2\mathbf{i} - \mathbf{j} \\ &= -\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}\end{aligned}$$

$\angle CAB$ given by

$$\mathbf{AC} \cdot \mathbf{AB} = |\mathbf{AC}||\mathbf{AB}| \cos(\angle CAB)$$

$$\begin{aligned}\mathbf{AC} \cdot \mathbf{AB} &= (0, -3, 1) \cdot (+1, -1, -1) \\ &= +3 - 1 \\ &= +2\end{aligned}$$

$|\mathbf{AC}| = \sqrt{0^2 + 9 + 1} = \sqrt{10}$
$|\mathbf{AB}| = \sqrt{1 + 1 + 1} = \sqrt{3}$
Therefore $\cos(\angle CAB) = \dfrac{+2}{\sqrt{10}\sqrt{3}} = \dfrac{+2}{\sqrt{30}} = 0.51639$
$\Rightarrow \angle CAB = \arccos\left(\dfrac{+2}{\sqrt{30}}\right) = 68.583°$
$\mathbf{BA} = -\mathbf{AB} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$; $\mathbf{BC} = -\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$
Therefore $\mathbf{BA} \cdot \mathbf{BC} = (-1, 1, 1)(-1, -2, 2) = 1 - 2 + 2 = 1$
$|\mathbf{AB}| = \sqrt{3}$
$|\mathbf{AB}| = \sqrt{1 + 4 + 4} = 3$
Therefore $\angle CBA = \arccos\left(\dfrac{1}{3\sqrt{3}}\right) = \arccos(0.19245) = 78.904°$
$\angle BCA = 180 - \arccos\left(\dfrac{2}{\sqrt{30}}\right) - \arccos\left(\dfrac{1}{3\sqrt{3}}\right) = 32.51$

(iii)

$$\begin{aligned}\text{Area of } \triangledown &= \frac{1}{2}|\mathbf{AB} \times \mathbf{AC}| \\ &= \frac{1}{2}\left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 0 & -3 & 1 \end{matrix} \right\|\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left|\mathbf{i}\begin{vmatrix} -1 & -1 \\ -3 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & -1 \\ 0 & -3 \end{vmatrix}\right| \\
&= \frac{1}{2}|-4\mathbf{i} - \mathbf{j} - 3\mathbf{k}| \\
&= \frac{1}{2}\sqrt{16+1+9} \\
&= \frac{\sqrt{26}}{2} = \sqrt{\frac{13}{2}} = 2.549
\end{aligned}
$$