Question

(i) Find the general solution of the equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0.$$

(ii) Use the result of part (i) to find the solution of

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 2e^{-3x},$$

$$\text{where } y = 1 \text{ and } \frac{dy}{dx} = -3 \text{ when } x = 0.$$

Answer

(i) Use trial function $y = Ae^{kx}$

$\Rightarrow \dfrac{dy}{dx} = Ake^{kx}, \ \dfrac{d^2y}{dx^2} = Ak^2e^{kx}$

Therefore auxiliary equation is

$$k^2 + 4k + 5 = 0$$

$$\begin{aligned} \Rightarrow k &= \frac{-4 \pm \sqrt{16 - 20}}{2} \\ &= -2 \pm \frac{2i}{2} \\ &= -2 \pm i \end{aligned}$$

Therefore general solution is of the form

$$y = Ae^{-2x+ix} + Be^{-2x-ix}$$

<u>or</u>

$$\underline{y = e^{-2x}(C\cos x + D\sin x)}$$

$C, \ D$ constants

(ii) This is the same equation as (i) but with forcing term $2e^{-3x}$ on RHS

Therefore

$$y = y_{CF} + y_{PI}$$

where y_{CF} satisfies

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$$

Thus from (i)

$$y_{CF} = e^{-2x}(C\cos x + D\sin x)$$

For y_{PI} try

$$y_{PI} = Ae^{-3x}$$

and substitute in full equation

$y'_{PI} = -3Ae^{-3x},\ y''_{PI} = +9Ae^{-3x}$

Therefore

$$\begin{aligned} 9Ae^{-3x} - 4 \times 3Ae^{-3x} + 5Ae^{-3x} &= 2e^{-3x} \\ \Rightarrow 2Ae^{-3x} &= 2e^{-3x} \\ \Rightarrow A &= 1 \end{aligned}$$

Therefore

$$\underline{y_{PI} = e^{-3x}}$$

Thus the general solution is

$$\underline{y = e^{-2x}(C\cos x + D\sin x) + e^{-3x}}$$

Specific solution from boundary conditions:

$y = 1,\ x = 0$

$\Rightarrow 1 = 1(C + 0) + 1$

$\Rightarrow C = 0$

Therefore $y = De^{-2x}\sin x + e^{-3x}$

So

$$\frac{dy}{dx} = D(-2e^{-2x}\sin x + e^{-2x}\cos x) - 3e^{-3x}$$

$$\frac{dy}{dx} = -3 \text{ when } x = 0$$

$$\Rightarrow -3 = D(0+1) - 3$$

$$\Rightarrow D = 0$$

Therefore $\underline{y = e^{-3x}}$ is the specific solution.