QUESTION

**(i)** Sketch the region defined by the inequalities:

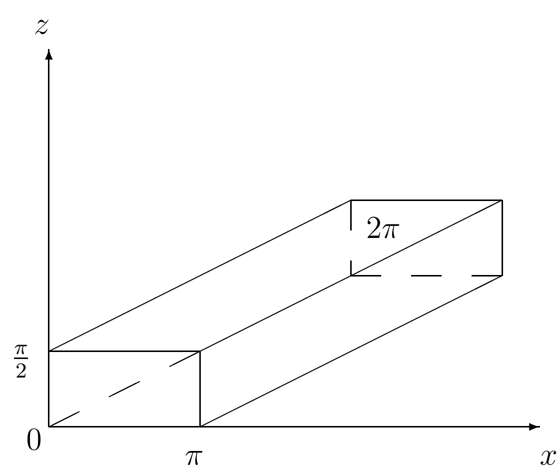
$$0 \leq x \leq \pi,\ 0 \leq y \leq 2\pi,\ 0 \leq z \leq \frac{\pi}{2}.$$

**(ii)** If the region is occupied by a solid $S$ with density at any point $(x, y, z)$ given by the formula $2xy^2 \cos z$, compute the total mass of the region $S$ by evaluating an appropriate triple integral.

**(iii)** The region $S$ is divided by the plane $x = ay$ (where $a$ is a constant $0 < a < \frac{1}{2}$) into two regions: the region $S_1$ contains the point $(\pi, 0, 0)$ and the region $S_2$ contains the point $(0, 2\pi, 0)$. Sketch the two regions $S_1$ and $S_2$, and find the mass of $S_1$ in terms of $a$.

**(iv)** Using your answers to parts (i) and (ii), find the mass of the upper part $S_2$, again in terms of $a$, and find the value of $a$ for which the two regions have equal mass.

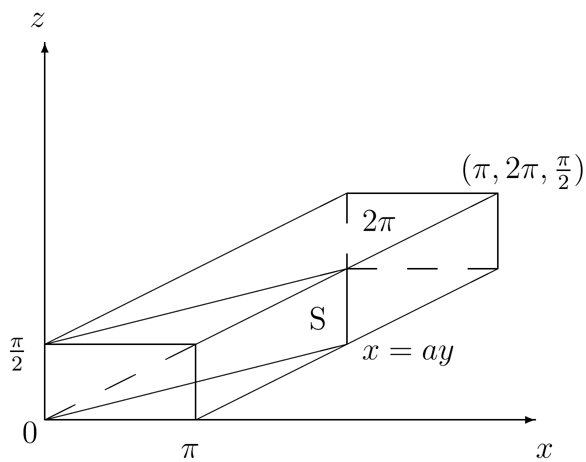
ANSWER




**(i)**

**(ii)**

$$\begin{aligned}
\text{Mass} &= \int_0^{2\pi} y^2\,dy \int_0^{\pi} 2x\,dx \int_0^{\frac{\pi}{2}} \cos z\,dz \\
&= \left[\frac{y^3}{3}\right]_0^{2\pi} \left[x^2\right]_0^{\pi} [\sin z]_0^{\frac{\pi}{2}} \\
&= \frac{8\pi^5}{3}
\end{aligned}$$




**(iii)**

$$\begin{aligned}
\text{Mass of } S_1 &= \int_{x=0}^{\pi}\int_{y=0}^{\frac{x}{a}}\int_{z=0}^{\frac{\pi}{2}} y^2 2x \cos z \, dz dy dx \\
&= \int_{x=0}^{\pi}\int_{y=0}^{\frac{x}{a}} [\sin z]_0^{\frac{\pi}{2}} \, y^2 2x \, dy dx \\
&= \int_{x=0}^{\pi}\int_{y=0}^{\frac{x}{a}} 2xy^2 \, dy dx \\
&= \int_{x=0}^{\pi} \left[2x\frac{y^3}{3}\right]_{y=0}^{\frac{x}{a}} dx \\
&= \int_0^{\pi} \frac{2x^4}{3a^3} \, dx \\
&= \left[\frac{2x^5}{15a^3}\right]_0^{\pi} = \frac{2\pi^5}{15a^3}
\end{aligned}$$

**(iv)** mass of $S_2 = \left(\frac{8}{3} - \frac{2}{15a^3}\right)\pi^5$

mass $(S_1)$ =mass $(S_2) \Leftrightarrow \frac{4}{15a^3} = \frac{8}{3} \Leftrightarrow a^3 = \frac{1}{10}$ or $a = \frac{1}{\sqrt[3]{10}}$