QUESTION

(i) Set up the following system of linear diffenrential equations as a matrix problem:

$$\frac{dx}{dt} + y + z = 3x \; \frac{dy}{dt} + 6y + z = x \; \frac{dz}{dt} + y = 2z.$$

(ii) Write down the general form of the solution to the problem.

(iii) Find the particular solution subject to the initial conditions $\frac{dx}{dt} = -218,\ \frac{dy}{dt} = -78,\ \frac{dz}{dt} = 52$ when $t = 0$.

ANSWER

(i)

$$D\begin{pmatrix} 3 & -1 & -1 \\ 1 & -6 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(ii) The general solution has form

$$k_1\mathbf{v}_1e^{\lambda_1 t} + k_2\mathbf{v}_2e^{\lambda_2 t} + k_3\mathbf{v}_3e^{\lambda_3 t}$$

where $\{\mathbf{v}_i\}$ = eigenvectors and $\{\lambda_i\}$ = eigenvalues of A

Solve $|A - \lambda I| = 0$ for λ

$$\begin{vmatrix} 3-\lambda & -1 & -1 \\ - & -6-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = (3-\lambda)\left[(-6-\lambda)(2-\lambda) - 1\right] - \left[-(2-\lambda) - 1\right]$$

$$= (3-\lambda)(-6-\lambda)(2-\lambda)$$

Eigenvalues $\lambda = -6, 2, 3$.

$\lambda_1 = -6$ solve $\begin{pmatrix} 9 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & -1 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$8x = y = 8z,\ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix}$

$$\lambda_2 = 2 \text{ solve } \begin{pmatrix} -1 & -1 & -1 \\ 1 & -8 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$y = 0,\ x = z,\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_3 = 3 \text{ solve } \begin{pmatrix} 0 & -1 & -1 \\ 1 & -9 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-y = z,\ x = 8y,\ \mathbf{v}_3 = \begin{pmatrix} 8 \\ 1 \\ -1 \end{pmatrix}$$

Solution is

$$\begin{aligned} x &= k_1 e^{-6t} + k_2 e^{2t} + 8k_3 e^{3t} \\ y &= 8k_1 e^{-6t} + k_3 e^{3t} \\ z &= k_1 e^{-6t} + k_2 e^{2t} - k_3 e^{3t} \end{aligned}$$

(iii) When $t = 0$,

$$D \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6k_1 & +2k_2 & +24k_3 \\ -48k_1 & & +3k_3 \\ -6k_1 & +2k_2 & -3k_3 \end{pmatrix}$$

Solve $\begin{pmatrix} -6 & 2 & 24 \\ -48 & 0 & 3 \\ -6 & 2 & -3 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -218 \\ -78 \\ 52 \end{pmatrix}$, the initial conditions.

row(1)-row(3)	$27k_3 = -270$	$k_3 = -10$
row(2)	$-48k_1 - 30 = -78$	$k_1 = 1$
row(3)	$-6 + 2k_2 + 30 = 52$	$k_2 = 14$

Therefore the particular solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} e^{-6t} + 14e^{2t} - 80e^{3t} \\ 8e^{-6t} - 10e^{3t} \\ e^{-6t} + 14e^{2t} + 10e^{3t} \end{pmatrix}$$