QUESTION

(i) Find the solution to the following differential equation given the condition $y(0) = 0$:

$$\frac{dy}{dx} = \sin x \cos^2 y.$$

(ii) Find the general solution to the differential equation. Note any values of x for which the solutions may not be valid:

$$x \log x \frac{dy}{dx} + y = 2 \log x.$$

[Hint: Note that $\frac{d}{dx}\{\log(\log x)\} = \frac{1}{x \log x}$.]

(iii) Find the solution to the following differential equation given the conditions: $y(0) = 1, y'(0) = 7$:

$$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 29y = 0.$$

ANSWER

(i) Separable

$$\begin{aligned} \int \frac{1}{\cos^2 y}\, dy &= \int \sin x\, dx \\ \tan y &= -\cos x + k \\ y(0) = 0 &\Rightarrow \tan(0) = -\cos(0) + k \\ &\Rightarrow 0 = k - 1 \text{ or } k = 1 \\ \tan y &= 1 - \cos x \end{aligned}$$

(ii) Integrating factor

$$\frac{dy}{dx} + \frac{1}{x \log x} y = \frac{2 \log x}{x \log x} \; x \neq 0, 1$$

$$IF = e^{\int \frac{1}{x \log x}\, dx} = e^{\log(\log x)} = \log x$$

so

$$y \log x = \int \frac{2 \log x}{x}\, dx = \log^2 x + k$$

by putting $u = \log x$

or

$$y = \frac{k}{\log x} + \log x,\ x \neq 1$$

This is valid for $x > o,\ x \neq 1$

(iii) The auxilliary equation is $\lambda^2 - 10\lambda + 29 = 0$ which has roots $\lambda = \frac{10 \pm \sqrt{100-116}}{2} = 5 \pm 2i$

The solution is therefore

$$y = e^{5x}(A \cos 2x + B \sin 2x)$$

$1 = y(0) = A,$

$y'(0) = 5e^{5x}(A \cos 2x + B \sin 2x) + e^{5x}(-2A \sin 2x + 2B \cos 2x)$

so $7 = y'(0) = 5A + 2B = 5 + 2B \Rightarrow B = 1$

Hence the solution is

$$y = e^{5x}(\cos 2x + \sin 2x)$$