Question

Verify that $f_{xy} = f_{yx}$ for:

(a) $f(x,y) = \exp\left(-(x^2+y^2-0.25)^2\right)$

(b) $f(x,y) = x^2\cos(x+2y) + y^2\cos(2x+y)$

Answer

(a)

$$\begin{aligned} h &= \exp\left(-(x^2+y^2-0.25)^2\right) \\ \frac{\partial h}{\partial x} &= -4x(x^2+y^2-0.25)\exp\left(-(x^2+y^2-0.25)^2\right) \\ \frac{\partial h}{\partial y} &= -4y(x^2+y^2-0.25)\exp\left(-(x^2+y^2-0.25)^2\right) \end{aligned}$$

$$\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial h}{\partial y}\right]$$

$$= \exp\left(-(x^2+y^2-0.25)^2\right)\left(-8xy + 4x(x^2+y^2-0.25)^2 \times 2 \times 2y\right)$$

$$\frac{\partial^2 h}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial h}{\partial x}\right]$$

$$= \exp\left(-(x^2+y^2-0.25)^2\right)\left(-8xy + 4x(x^2+y^2-0.25)^2 \times 2 \times 2y\right)$$

So $\dfrac{\partial^2 h}{\partial x \partial y} = \dfrac{\partial^2 h}{\partial y \partial x}$ as expected

(b)

$$\begin{aligned} f(x,y) &= x^2\cos(x+2y) + y^2\cos(2x+y) \\ \frac{\partial f}{\partial x} &= 2x\cos(x+2y) - x^2\sin(x+2y) - 2y^2\sin(2x+y) \\ \frac{\partial f}{\partial y} &= -2x^2\sin(x+2y) - 2y\cos(2x+y) + y^2\sin(2x+y) \end{aligned}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$$

$$= -4x\sin(x+2y) - 2x^2\cos(x+2y) - 4y\sin(2x+y) - 2y^2\cos(2x+y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$$

$= -4x\sin(x+2y) - 2x^2\cos(x+2y) - 4y\sin(2x+y) - 2y^2\cos(2x+y)$

So $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$ as expected